



# A tree approach to $p$ -variation and to integration

Jean Picard

## ► To cite this version:

Jean Picard. A tree approach to  $p$ -variation and to integration. Annals of Probability, 2008, 36 (6), pp.2235-2279. 10.1214/07-AOP388 . hal-00146698v2

**HAL Id: hal-00146698**

**<https://hal.science/hal-00146698v2>**

Submitted on 22 Jan 2009

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## A TREE APPROACH TO $P$ -VARIATION AND TO INTEGRATION

BY JEAN PICARD

*Université Blaise Pascal*

We consider a real-valued path; it is possible to associate a tree to this path, and we explore the relations between the tree, the properties of  $p$ -variation of the path, and integration with respect to the path. In particular, the fractal dimension of the tree is estimated from the variations of the path, and Young integrals with respect to the path, as well as integrals from the rough paths theory, are written as integrals on the tree. Examples include some stochastic paths such as martingales, Lévy processes and fractional Brownian motions (for which an estimator of the Hurst parameter is given).

**1. Introduction.** Consider a continuous path  $\omega : [0, 1] \rightarrow \mathbb{R}$ . The  $p$ -variation of  $\omega$  is defined for  $p \geq 1$  by

$$V_p(\omega) := \sup_{(t_i)} \sum_i |\omega(t_{i+1}) - \omega(t_i)|^p$$

for subdivisions  $(t_i)$  of  $[0, 1]$ . It is well known that the finiteness of  $V_p(\omega)$  is closely related to the possibility of constructing integrals  $\int_0^1 \rho d\omega$  for some functions  $\rho$ . The simplest case is when  $V_1(\omega)$  is finite ( $\omega$  has finite variation); then a signed measure  $d\omega = d\omega^+ - d\omega^-$  (the Lebesgue–Stieltjes measure) is defined from  $\omega$ , and the integral is well defined for any bounded Borel function  $\rho$ ; if moreover  $\rho$  has left and right limits, then the integral is also a Riemann–Stieltjes integral (it is the limit of Riemann sums). If now  $\omega$  has infinite variation ( $V_1(\omega) = \infty$ ) but  $V_p(\omega)$  is finite for a larger value of  $p$ , it was proved by Young [36] that a Riemann–Stieltjes integral can still be constructed as soon as  $V_q(\rho)$  is finite for  $q$  such that  $1/p + 1/q > 1$ ; as an application, one can consider and solve stochastic differential equations driven by a multidimensional path with finite  $p$ -variation if  $p < 2$  (in particular a typical fractional Brownian path with Hurst parameter  $H > 1/2$ ). If now  $p$

---

Received May 2007; revised December 2007.

*AMS 2000 subject classifications.* 60G17, 60H05, 26A42.

*Key words and phrases.* Lebesgue–Stieltjes integrals, rough paths, real trees, variations of paths, fractional Brownian motion, Lévy processes.

This is an electronic reprint of the original article published by the Institute of Mathematical Statistics in *The Annals of Probability*, 2008, Vol. 36, No. 6, 2235–2279. This reprint differs from the original in pagination and typographic detail.

is greater than 2, Lyons's theory of rough paths [20, 21, 22, 23] provides a richer framework which is still suitable to consider and solve these equations.

On the other hand, one can associate to  $\omega$  a metric space  $(\mathbb{T}, \delta)$  which is a compact real tree and which can be used to describe the excursions of  $\omega$  above any level; see [6, 8] or Chapter 3 of [10]. The tree  $\mathbb{T}$  can be endowed with its length measure  $\lambda$ , and our aim is to relate the properties of  $(\mathbb{T}, \delta, \lambda)$  to the questions of  $p$ -variation of  $\omega$  and of integration with respect to  $\omega$ . These questions are also considered for càdlàg paths  $\omega$  (paths which are right-continuous and have left limits), since these paths can be considered as time-changed continuous paths. As an application, we consider the case where  $\omega$  is a path of a stochastic process such as a Lévy process or a fractional Brownian motion (the case of a standard Brownian path has been considered in [30]).

In Section 2, we introduce the tree  $\mathbb{T}$  and study its basic properties. In particular, in the finite variation case, we work out the interpretation of its length measure  $\lambda$  by means of the Lebesgue–Stieltjes measure of  $\omega$ , extending a result of [6]; this result is fundamental for the construction of integrals in Section 4 (see below). We also explain how the tree can be defined in the càdlàg case.

In Section 3, we see in Theorem 3.1 (Theorem 3.10 for the càdlàg case) that the finiteness of  $V_p(\omega)$  is related to some metric properties of  $\mathbb{T}$ , particularly its upper box dimension  $\overline{\dim} \mathbb{T}$ ; more precisely,

$$(1.1) \quad \begin{cases} V_p(\omega) = \infty, & \text{if } 1 \leq p \leq \overline{\dim} \mathbb{T}, \\ V_p(\omega) < \infty, & \text{if } p > \overline{\dim} \mathbb{T}. \end{cases}$$

We give applications of these results to martingales, fractional Brownian motions and Lévy processes. We prove in particular that upper box and Hausdorff dimensions of  $\mathbb{T}$  coincide for fractional Brownian motions (with Hurst parameter  $H$ ) and stable Lévy processes (with index  $\alpha$ ); we also construct an estimator of  $H$  based on  $\mathbb{T}$ , which can be computed by means of a sequence of stopping times (Proposition 3.9).

The aim of Section 4 is to construct integrals with respect to  $\omega$  by means of the tree. Let us assume that  $\omega$  is continuous and  $\omega(0) = \omega(1) = \inf \omega$  (considering the general case adds some notational complication). The construction of the integral is based on the following remark (Propositions 2.2 and 2.3): when  $\omega$  has finite variation, the positive and negative parts  $d\omega^+$  and  $d\omega^-$  of  $d\omega$  can be viewed as the images of the length measure  $\lambda$  by two maps  $\tau \mapsto \tau^{\nearrow}$  and  $\tau \mapsto \tau^{\searrow}$  from  $\mathbb{T}$  to  $[0, 1]$ ; thus

$$(1.2) \quad \int_0^1 \rho d\omega = \int_{\mathbb{T}} (\rho(\tau^{\nearrow}) - \rho(\tau^{\searrow})) \lambda(d\tau).$$

When  $\omega$  has infinite variation, this procedure can still be applied to construct  $d\omega^+$  and  $d\omega^-$ ; these measures are  $\sigma$ -finite but no more finite. However, (1.2)

can be viewed as a definition of  $\int \rho d\omega$  provided the term in the right-hand side is integrable; this means that the tree can provide a mechanism by means of which  $d\omega^+$  and  $d\omega^-$  compensate each other. For instance, if  $1/p + 1/q > 1$ ,

$$V_p(\omega) < \infty, \quad V_q(\rho) < \infty \quad \implies \quad \int_{\mathbb{T}} |\rho(\tau^{\nearrow}) - \rho(\tau^{\searrow})| \lambda(d\tau) < \infty.$$

Moreover, in this case, the integral defined by (1.2) coincides with the Young integral (Theorems 4.1 and 4.5). Consequently, differential equations driven by multidimensional paths with finite  $p$ -variation with  $p < 2$  enter our framework. Actually, we may take  $p > 2$  for one of the components (Theorem 4.8); this is due to the fact that the condition  $V_q(\rho) < \infty$  can be replaced by some weaker condition  $V_q(\rho|\omega) < \infty$ . We also prove that the tree approach can be used to consider multidimensional fractional Brownian motions with parameter  $H > 1/3$  (Theorem 4.9); in this case, the right-hand side of (1.2) should be understood as a generalized integral on  $\mathbb{T}$  (a limit of integrals on subtrees  $\mathbb{T}^a$  obtained by trimming  $\mathbb{T}$ ), and we recover the integrals of the rough paths theory.

The Appendix is devoted to two results which are needed in the article, and which may also be of independent interest. In Appendix A.1, we prove that increments of fractional Brownian motions are asymptotically independent from the past. In Appendix A.2, we study the time discretization of integrals in the rough paths calculus, in a spirit similar to [13, 15].

REMARK 1.1. A lot of work has been devoted to the links between random trees and excursions of some stochastic processes; these links are an extension of the classical Harris correspondence between random walks and random finite trees. Historically, they have first been investigated in the context of Brownian excursions in [1, 18, 26] (see also the courses [10, 32]) with the aim of studying branching processes. In order to consider more general branching mechanisms, Lévy trees, defined by means of Lévy processes  $X$  without negative jumps, have been introduced and studied in [7, 19]; they have been related to the notion of real tree in [8]. However, we will not focus here on properties of Lévy trees; a Lévy tree is indeed a tree which is associated to some continuous process related to  $X$  (the height process), whereas we will rather consider in our applications the tree which is associated directly to the Lévy process  $X$ .

REMARK 1.2. We work out here a nonlinear approach to integration with respect to one-dimensional paths; consequently, the integral with respect to  $\omega_1 + \omega_2$  is not simply related to integrals with respect to  $\omega_1$  and  $\omega_2$ ; moreover, integration with respect to a multidimensional path can be worked out by summing integrals with respect to each component, but this depends on the choice of a frame.

REMARK 1.3. In the proofs of this article, the letter  $C$  will denote constant numbers which may change from line to line. For quantities depending on the path  $\omega$  of a stochastic process, we will rather use the notation  $K = K(\omega)$ .

**2. Paths and trees.** In this section, we first define the tree associated to a continuous path, describe its length measure, and extend these objects to càdlàg paths.

2.1. *Basic definitions and properties.* Consider a continuous function  $(\omega(t); 0 \leq t \leq 1)$ . The function

$$(2.1) \quad \delta(s, t) := \omega(s) + \omega(t) - 2 \inf_{[s, t]} \omega$$

is a semi-distance on  $[0, 1]$ , where

$$\delta(s, t) = 0 \iff \omega(s) = \omega(t) = \inf_{[s, t]} \omega.$$

The quotient metric space  $\mathbb{T} = ([0, 1]/\delta, \delta)$  is a real tree; this means that between any two points  $\tau_1$  and  $\tau_2$  in  $\mathbb{T}$ , there is a unique arc denoted by  $[\tau_1, \tau_2]$  ( $\mathbb{T}$  is a topological tree), and that  $[\tau_1, \tau_2]$  is isometric to the interval  $[0, \delta(\tau_1, \tau_2)]$  of  $\mathbb{R}$ ; see [8]. Actually, real trees can also be characterized as connected metric spaces satisfying the so-called four-point condition, and one can use this condition to prove that  $\mathbb{T}$  is a real tree; see [6, 10]. We will denote by  $\pi$  the projection of  $[0, 1]$  onto  $\mathbb{T}$ ; notice that if  $\omega$  is constant on some interval  $[s, t]$ , then all the points of this interval are projected on the same point of  $\mathbb{T}$ . The continuity of  $\pi$  follows from the continuity of  $\omega$ ; in particular,  $\mathbb{T}$  is compact. In this article we implicitly assume that  $\omega$  is not constant, so that  $\mathbb{T}$  is not reduced to a singleton.

We now suppose  $\pi(0) = \pi(1)$ , or equivalently

$$(2.2) \quad \omega(0) = \omega(1) = \inf_{[0, 1]} \omega.$$

An example is given in Figure 1. We explain at the end of the subsection how general paths can be reduced to this case. Under this condition,  $\mathbb{T}$  becomes a rooted tree by considering  $\pi(0) = \pi(1) = O$  as the root of the tree, and we can say that a point  $\tau_1$  is above  $\tau_2$  if  $\tau_2 \in [O, \tau_1]$ .

We consider on  $\mathbb{T}$  the level function  $\ell$  defined by

$$(2.3) \quad \ell(\tau) := \omega(0) + \delta(O, \tau).$$

Then  $\omega = \ell \circ \pi$ . For  $\tau$  in  $\mathbb{T}$ , define

$$\tau^{\nearrow} := \inf \pi^{-1}(\tau), \quad \tau^{\searrow} := \sup \pi^{-1}(\tau),$$

so that

$$\omega(\tau^{\nearrow}) = \omega(\tau^{\nwarrow}) = \inf_{[\tau^{\nearrow}, \tau^{\nwarrow}]} \omega = \ell(\tau).$$

In particular  $O^{\nearrow} = 0$  and  $O^{\nwarrow} = 1$ . The set  $\pi([\tau^{\nearrow}, \tau^{\nwarrow}])$  is exactly the set of points above  $\tau$ . If now we consider the set  $\pi([\tau^{\nearrow}, \tau^{\nwarrow}]) \setminus \{\tau\}$  of points which are strictly above  $\tau$ , it is made of connected components which are subtrees, and which are called the branches above  $\tau$ ; each of these branches is the projection of a connected component of  $[\tau^{\nearrow}, \tau^{\nwarrow}] \setminus \pi^{-1}(\tau)$ , and corresponds to an excursion of  $\omega$  above level  $\ell(\tau)$ . If there is more than one branch above  $\tau$ , then  $\tau$  is said to be a branching point; this means that there is more than one excursion, and the times between these excursions are local minima of  $\omega$  (a local minimum may be a constancy interval). On the other hand, if there is no branch above  $\tau$ , then  $\tau$  is said to be a leaf; this means that  $\pi^{-1}(\tau) = [\tau^{\nearrow}, \tau^{\nwarrow}]$ , so this holds when  $\tau^{\nearrow} = \tau^{\nwarrow}$  or when  $[\tau^{\nearrow}, \tau^{\nwarrow}]$  is a constancy interval of  $\omega$ . Local maxima of  $\omega$  are projected on leaves of  $\mathbb{T}$ , but there may be leaves which are not associated to local maxima. Points which are not leaves constitute the skeleton  $S(\mathbb{T})$  of the tree.

We say that  $\omega$  is piecewise monotone if there exists a finite subdivision  $(t_i)$  of  $[0, 1]$  such that  $\omega$  is monotone on each  $[t_i, t_{i+1}]$ . We also say that  $\mathbb{T}$  is finite if it has finitely many leaves. If  $\mathbb{T}$  is not finite, then it has infinitely many branching points, or it has at least a branching point with infinitely

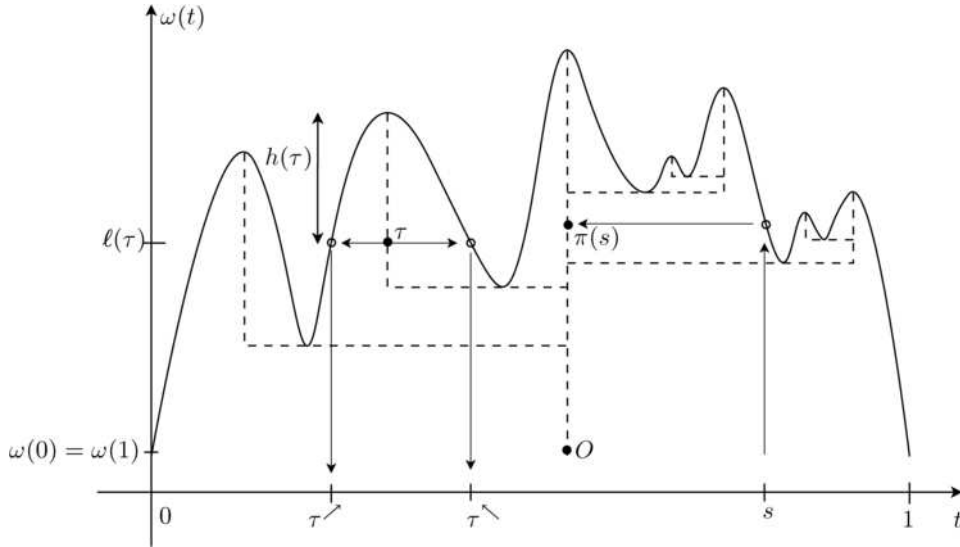


FIG. 1. An example of path  $\omega$  with its tree  $\mathbb{T}$  represented by dashed lines (the vertical lines represent points of the skeleton, and each branching point is represented by a horizontal line); maps  $\tau \mapsto \tau^{\nearrow}$ ,  $\tau \mapsto \tau^{\nwarrow}$  and  $s \mapsto \pi(s)$  are also depicted.

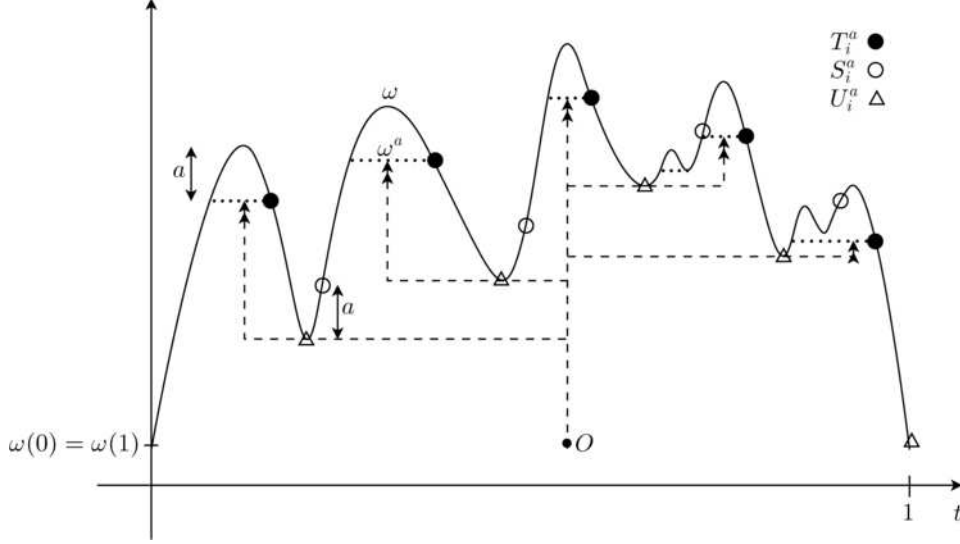


FIG. 2. The trimmed tree  $\mathbb{T}^a$  is represented by dashed lines, and its leaves by double arrows; the flattened path  $\omega^a$  is represented by dots when it differs from  $\omega$ ; times  $T_i^a$ ,  $S_i^a$  and  $U_i^a$ ,  $i \geq 1$ , are respectively represented on the curve by bullets, circles and triangles.

many branches above it; in both of these cases,  $\omega$  has infinitely many local minima and is therefore not piecewise monotone. Conversely, if  $\omega$  is not piecewise monotone, then it has infinitely many local maxima, and each of them is projected on a different leaf of  $\mathbb{T}$ , so  $\mathbb{T}$  is not finite. Thus

$$(2.4) \quad \omega \text{ is piecewise monotone} \iff \mathbb{T} \text{ is finite.}$$

We shall also need an operation called trimming, or leaf erasure, due to [25] (see also [10, 11, 17, 26]); to this end, we introduce the function

$$(2.5) \quad h(\tau) := \sup\{\omega(t) - \ell(\tau); \tau^{\nearrow} \leq t \leq \tau^{\searrow}\}.$$

This is the height of the (or of the highest) branch above  $\tau$ . In particular,  $h(\tau) = 0$  if and only if  $\tau$  is a leaf.

Now consider the trimmed tree

$$(2.6) \quad \mathbb{T}^a := \{\tau \in \mathbb{T}; h(\tau) \geq a\}.$$

Then  $\mathbb{T}^a$  is nonempty if and only if  $\|\omega\| := \sup \omega - \inf \omega \geq a$ , and in this case, it is a rooted subtree of  $\mathbb{T}$  (it contains the root  $O$ ). An example is drawn in Figure 2. As  $a \downarrow 0$ , the tree  $\mathbb{T}^a$  increases to the skeleton of  $\mathbb{T}$ ; each branch grows at unit speed, and a new branch appears at  $\tau$  if  $\tau$  is a branching point of  $\mathbb{T}$  such that one of the branches above  $\tau$  has height exactly  $a$ , and another one has height at least  $a$ . This subtree has been introduced in [26] and is

related to  $a$ -minima and  $a$ -maxima of the path. More precisely, starting with  $S_0^a = T_0^a = 0$ , define

$$(2.7) \quad \begin{cases} T_{i+1}^a := \inf \left\{ t \in [S_i^a, 1]; \omega(t) - \sup_{[S_i^a, t]} \omega < -a \right\}, \\ S_{i+1}^a := \inf \left\{ t \in [T_{i+1}^a, 1]; \omega(t) - \inf_{[T_{i+1}^a, t]} \omega > a \right\}, \\ N^a := \inf \{ i; T_i^a \text{ or } S_i^a = \inf \emptyset \}. \end{cases}$$

Actually, in the case  $\pi(0) = \pi(1)$ ,  $T_{N^a}^a$  is still well defined, but not  $S_{N^a}^a$  (notice in particular that if  $\omega$  is a path of an adapted stochastic process, then  $S_i^a$  and  $T_i^a$  are stopping times). Then  $N^a$  is the number of leaves of  $\mathbb{T}^a$ ; the set of leaves  $\partial \mathbb{T}^a$  and the set of times  $(T_i^a; 1 \leq i \leq N^a)$  are in bijection by means of  $\pi$  and its inverse map  $\tau \mapsto \tau^\setminus$ . Moreover

$$(2.8) \quad \inf_{[T_i^a, T_{i+1}^a]} \omega = \inf_{[T_i^a, S_i^a]} \omega = \omega(S_i^a) - a \quad \text{for } 1 \leq i < N^a.$$

The approximation of  $\mathbb{T}$  by  $\mathbb{T}^a$  can also be interpreted as an approximation of the path  $\omega$ ; trimming the tree is equivalent to flattening some excursions of the path. More precisely, let  $\pi^a(t)$  be the projection of  $\pi(t)$  on  $\mathbb{T}^a$  (assuming  $\mathbb{T}^a \neq \emptyset$ ), and let

$$(2.9) \quad \omega^a = \ell \circ \pi^a$$

for the level function  $\ell$  defined in (2.3). Then  $\mathbb{T}^a$  is the associated tree of  $\omega^a$ . The path  $\omega^a$  is continuous, is obtained from  $\omega$  by means of the change of time

$$\omega^a(t) = \omega(\inf \{ u \geq t; \pi(u) \in \mathbb{T}^a \}),$$

and satisfies  $0 \leq \omega - \omega^a \leq a$ . Since  $\mathbb{T}^a$  is finite, it follows from (2.4) that  $\omega^a$  is piecewise monotone. Actually, if  $U_i^a$  is a time of  $[T_i^a, S_i^a]$  at which  $\omega$  is minimal (for  $1 \leq i < N^a$ ) and if  $U_0^a := 0$ ,  $U_{N^a}^a := 1$ , then

$$(2.10) \quad \omega(U_i^a) = \omega(S_i^a) - a \quad \text{for } 1 \leq i < N^a,$$

and

$$(2.11) \quad \begin{cases} \omega^a \text{ is nondecreasing on } [U_i^a, T_{i+1}^a], \\ \omega^a \text{ is nonincreasing on } [T_i^a, U_i^a]. \end{cases}$$

Consider now a general continuous map  $\omega$  which does not satisfy  $\pi(0) = \pi(1)$ . Then we can again associate the tree  $\mathbb{T}$  by means of  $\delta$  defined by (2.1), but some of the above properties differ. However, it is still possible to apply the above discussion to an extended path  $\omega'$  defined on a greater interval, say  $[-1, 2]$ , coinciding with  $\omega$  on  $[0, 1]$ , and satisfying  $\omega'(-1) = \omega'(2) = \inf_{[-1, 2]} \omega'$ . Then the associated tree  $\mathbb{T}'$  contains  $\mathbb{T}$  as a subtree, and the projection  $\pi: [0, 1] \rightarrow \mathbb{T}$  is the restriction of  $\pi': [-1, 2] \rightarrow \mathbb{T}'$  to  $[0, 1]$ .



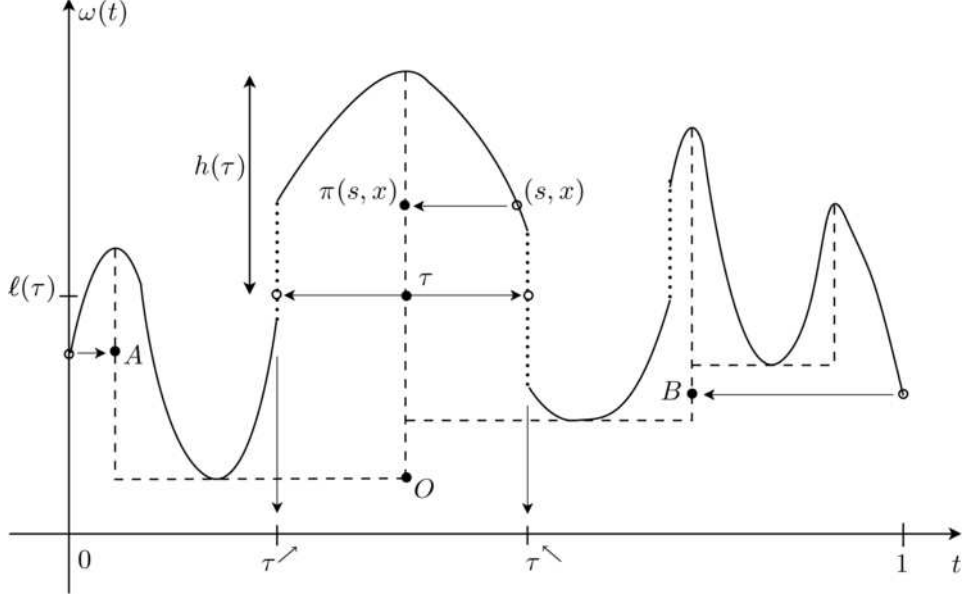


FIG. 3. A path with jumps, and its tree (dashed lines). The graph  $\mathbb{G}$  is the curve augmented by the jumps (dotted lines). Are also depicted the map  $\pi$  from  $\mathbb{G}$  to  $\mathbb{T}$ , the maps  $\tau \mapsto \tau^{\nearrow}, \tau \mapsto \tau^{\searrow}$  from  $\mathbb{T}$  to  $[0, 1]$ ; in particular,  $A = \pi(0, \omega(0))$  and  $B = \pi(1, \omega(1))$ .

Among these paths, we will only consider the minimal extensions; they are those such that  $\mathbb{T}' = \mathbb{T}$ . This means that

$$(2.12) \quad \begin{cases} \omega'(-1) = \omega'(2) = \inf_{[0,1]} \omega, \\ \omega' \text{ is nondecreasing on } [-1, 0], \text{ nonincreasing on } [1, 2]. \end{cases}$$

Let  $U$  be a time of  $[0, 1]$  at which  $\omega$  is minimal and consider

$$(2.13) \quad O := \pi(U), \quad A := \pi(0), \quad B := \pi(1)$$

(these points are drawn in Figure 3 below, in the more general case of paths with jumps). We choose  $O$  as the root of  $\mathbb{T}$ . Then  $O$  belongs to  $[A, B]$ , the points of  $[O, A]$  are those such that  $\tau^{\nearrow} \leq 0 \leq \tau^{\searrow} \leq 1$ , and the points of  $[O, B]$  are those such that  $0 \leq \tau^{\nearrow} \leq 1 \leq \tau^{\searrow}$ ; for the points of  $\mathbb{T} \setminus [A, B]$ , one has  $0 < \tau^{\nearrow} \leq \tau^{\searrow} < 1$ .

In particular, if we trim the tree  $\mathbb{T}$  and if  $\mathbb{T}^a \neq \emptyset$ , then the flattened path  $\omega^a$  of (2.9) is the restriction of  $\omega'^a$  to  $[0, 1]$ . Moreover, the quantities  $N^a, T_i^a$  and  $S_i^a$  defined in (2.7) and the similar quantities for  $\omega'$  satisfy

$$N^a = N'^a, \quad S_i^a = (S')_i^a, \quad T_i^a = (T')_i^a \quad \text{for } 1 \leq i < N^a.$$

At  $i = N^a$ , the time  $(T')_{N^a}^a$  may be after time 1, and in this case  $T_{N^a}^a$  is not defined.

**2.2. The length measure on the tree.** The length measure on  $\mathbb{T}$  is the unique measure  $\lambda$  which is supported by the skeleton (the set of leaves have zero measure) and such that the measure of an arc is equal to its length; in particular, this measure is  $\sigma$ -finite and atomless. The existence and uniqueness of  $\lambda$  is elementary for the finite subtrees  $\mathbb{T}^a$ , and it is not difficult to deduce the result for  $\mathbb{T}$  by letting  $a \downarrow 0$ . It can be identified to either of the two following measures.

**PROPOSITION 2.1.** *Define*

$$\lambda_1 := \int_{\mathbb{R}} \sum_{\tau \in S(\mathbb{T}) : \ell(\tau) = x} \delta_{\tau} dx, \quad \lambda_2 := \int_0^{\infty} \sum_{\tau \in \partial \mathbb{T}^a} \delta_{\tau} da = \int_0^{\infty} \sum_{\tau : h(\tau) = a} \delta_{\tau} da,$$

where  $\delta_{\tau}$  denotes the Dirac mass at  $\tau$ . Then  $\lambda = \lambda_1 = \lambda_2$ .

Notice that the number of terms in the sum is at most countable for any  $x$  in the definition of  $\lambda_1$ , whereas it is finite for any  $a > 0$  in the definition of  $\lambda_2$ . The integrals are supported by the interval  $[\inf \omega, \sup \omega]$  for the first one, and  $[0, \sup \omega - \inf \omega]$  for the second one.

**PROOF OF PROPOSITION 2.1.** The two measures are supported by the skeleton of the tree; in order to check that they are equal to  $\lambda$ , it is sufficient to verify that they coincide with it on arcs  $[O, \tau]$  for any  $\tau$  in the skeleton  $S(\mathbb{T})$ . The maps  $\ell$  and  $h$  are injective on  $[O, \tau]$ , so, if  $\lambda_{\mathbb{R}}$  denotes the Lebesgue measure on  $\mathbb{R}$ ,

$$\lambda_1([O, \tau]) = \lambda_{\mathbb{R}}(\ell([O, \tau])), \quad \lambda_2([O, \tau]) = \lambda_{\mathbb{R}}(h([O, \tau])).$$

Moreover,  $\ell$  induces a bijection between  $[O, \tau]$  and  $[\ell(O), \ell(\tau)]$ , so

$$\lambda_1([O, \tau]) = \ell(\tau) - \ell(O) = \delta(O, \tau) = \lambda([O, \tau]).$$

Thus  $\lambda_1 = \lambda$ . For the study of  $\lambda_2$ , notice that  $h(\tau_0)$  is the distance between  $\tau_0$  and any of the highest points above it. When  $\tau_0$  goes from  $O$  to  $\tau$ , then  $h(\tau_0)$  is decreasing; more precisely, it jumps at  $\tau_0$ , when  $\tau_0$  is a branching point so that no highest point above it is in the direction of  $\tau$ ; thus  $h$  has a finite number of negative jumps, and between these jumps, it is affine with slope  $-1$ . Consequently,  $h$  induces a bijection from  $[O, \tau]$  onto its image, and this image has Lebesgue measure  $\delta(O, \tau)$ . We deduce that  $\lambda_2 = \lambda$ .  $\square$

The measure  $\lambda$  is closely related to the two following measures on  $[0, 1]$ . Say that an excursion begins at time  $t$  above level  $\omega(t)$  if for some  $\varepsilon > 0$ ,  $\omega(s) > \omega(t)$  for  $t < s < t + \varepsilon$ . Let  $E^{\nearrow}$  be the set of beginnings of excursions above any level; we can define similarly the set  $E^{\searrow}$  of ends of excursions. These two sets are in bijection with each other; to each beginning  $t$  of an

excursion we can associate its end  $\inf\{s > t; \omega(s) = \omega(t)\}$ . If we restrict ourselves to a fixed level  $x$ , the sets of beginnings and ends of excursions above  $x$  are at most countable, and we can define

$$(2.14) \quad \omega^{\nearrow} := \int \sum_{s \in E^{\nearrow}; \omega(s)=x} \delta_s dx, \quad \omega^{\nwarrow} := \int \sum_{s \in E^{\nwarrow}; \omega(s)=x} \delta_s dx.$$

**PROPOSITION 2.2.** *Assume (2.2). The measures  $\omega^{\nearrow}$  and  $\omega^{\nwarrow}$  are  $\sigma$ -finite and are respectively the images of  $\lambda$  by the maps  $\tau \mapsto \tau^{\nearrow}$  and  $\tau \mapsto \tau^{\nwarrow}$ , and  $\lambda$  is the image of  $\omega^{\nearrow}$  and  $\omega^{\nwarrow}$  by the projection  $\pi$ . If (2.2) does not hold, then, with the notation (2.13), the maps  $\tau \mapsto \tau^{\nearrow}$  and  $\tau \mapsto \tau^{\nwarrow}$  are respectively defined on  $\mathbb{T} \setminus [O, A]$  and  $\mathbb{T} \setminus [O, B]$ ; the relation between  $\omega^{\nearrow}$  and  $\lambda$  (or between  $\omega^{\nwarrow}$  and  $\lambda$ ) again holds by restricting  $\lambda$  to  $\mathbb{T} \setminus [O, A]$  (or  $\mathbb{T} \setminus [O, B]$ ).*

**PROOF.** We only work out the proof under (2.2); the general case is easily deduced by considering an extension of  $\omega$  satisfying (2.12). We want to compare the measure  $\omega^{\nearrow}$  carried by the set  $E^{\nearrow}$  of beginnings of excursions, with the measure  $\lambda$  carried by the skeleton  $S(\mathbb{T})$ . If  $s$  is in  $E^{\nearrow}$ , then  $\pi(s)$  is in  $S(\mathbb{T})$  and  $s = \pi(s)^{\nearrow}$  except if  $s$  is at a local minimum, or the end of a constancy interval of  $\omega$ ; on the other hand, if  $\tau$  is in  $S(\mathbb{T})$ , then  $\tau = \pi(\tau^{\nearrow})$  and  $\tau^{\nearrow}$  is in  $E^{\nearrow}$  except if it is the beginning of a constancy interval of  $\omega$ . Since there are at most countably many local minima and constancy intervals, we deduce that there exists  $E_0^{\nearrow} \subset E^{\nearrow}$  and  $S_0(\mathbb{T}) \subset S(\mathbb{T})$  such that  $E^{\nearrow} \setminus E_0^{\nearrow}$  and  $S(\mathbb{T}) \setminus S_0(\mathbb{T})$  are at most countable, and the maps  $\tau \mapsto \tau^{\nearrow}$  and  $\pi$  are inverse bijections between  $E_0^{\nearrow}$  and  $S_0(\mathbb{T})$ . Moreover,  $\lambda$  and  $\omega^{\nearrow}$  are atomless, so they are supported respectively by  $S_0(\mathbb{T})$  and  $E_0^{\nearrow}$ . Thus the relation between  $\lambda$  and  $\omega^{\nearrow}$  claimed in the proposition follows from this one-to-one property, the definition (2.14) of  $\omega^{\nearrow}$  and the property  $\lambda = \lambda_1$  of Proposition 2.1. The case of  $\omega^{\nwarrow}$  is similar, and the  $\sigma$ -finiteness follows from the  $\sigma$ -finiteness of  $\lambda$ .  $\square$

We now give a condition on  $\mathbb{T}$  with which one can decide whether  $\omega$  has finite or infinite variation (this characterization is also given in [6]).

**PROPOSITION 2.3.** *The measures  $\lambda$ ,  $\omega^{\nearrow}$  and  $\omega^{\nwarrow}$  are finite if and only if  $\omega$  has finite variation. In this case,  $\omega^{\nearrow}$  and  $\omega^{\nwarrow}$  are respectively the positive and negative parts of the Lebesgue–Stieltjes measure of  $\omega$ . Moreover,*

$$(2.15) \quad \int_0^1 |d\omega| = 2\lambda(\mathbb{T}) - \delta(0, 1).$$

**PROOF.** We first work out the proof under the condition (2.2), so that  $\delta(0, 1) = 0$ . Suppose also that  $\mathbb{T}$  is finite, so that  $\lambda$  is finite and  $\omega$  is piecewise

monotone [as explained in (2.4)]. If, for instance,  $\omega$  is nondecreasing on  $[t_1, t_2]$ , then it is easily checked from the definitions (2.14) that

$$\omega^{\nearrow}([t_1, t_2]) = \omega(t_2) - \omega(t_1), \quad \omega^{\nwarrow}([t_1, t_2]) = 0.$$

A similar result holds for intervals on which  $\omega$  is nonincreasing, so we deduce that the proposition holds true in this case. If  $\mathbb{T}$  is not finite, consider the tree  $\mathbb{T}^a$  of (2.6) and its path  $\omega^a$  of (2.9). Notice that  $\lambda(\mathbb{T}^a) \uparrow \lambda(\mathbb{T})$  as  $a \downarrow 0$ . For  $b < a$ , one has  $\mathbb{T}^a \subset \mathbb{T}^b$ , and the path  $\omega^a$  is obtained from  $\omega^b$  by a change of time, so the variation of  $\omega^a$  increases as  $a$  decreases, and is bounded by the variation of  $\omega$ ; since the variation is a lower semicontinuous function of the path, it follows that the variation of  $\omega^a$  converges to the variation of  $\omega$  as  $a \downarrow 0$ , so

$$(2.16) \quad \int_0^1 |d\omega| = \lim \int_0^1 |d\omega^a| = 2 \lim \lambda(\mathbb{T}^a) = 2\lambda(\mathbb{T})$$

(we have applied the first part of the proof to  $\omega^a$  and  $\mathbb{T}^a$ ). Thus  $\omega$  has finite variation if and only if  $\lambda$  is finite. Moreover, if  $\omega$  has finite variation, one checks similarly that the positive part  $d\omega^+$  of the Lebesgue–Stieltjes measure of  $\omega$  satisfies

$$(d\omega^+)([s, t]) = \lim (d\omega^a)^+([s, t]) = \lim \omega^{\nearrow}([s, t] \cap \pi^{-1}(\mathbb{T}^a)) = \omega^{\nearrow}([s, t])$$

where we have used the fact that  $(\omega^a)^{\nearrow}$  is the restriction of  $\omega^{\nearrow}$  to  $\pi^{-1}(\mathbb{T}^a)$ . If (2.2) does not hold, we can consider an extension of  $\omega$  satisfying (2.12) and then restrict to  $[0, 1]$ . In this case, with the notation (2.13), the points  $\tau$  of  $[A, B]$  are such that  $\tau^{\nearrow} \leq 0$  or  $\tau^{\nwarrow} \leq 1$  and should not be counted twice in the total variation of  $\omega$  in (2.16). The correction which has to be made is  $\lambda([A, B]) = \delta(0, 1)$ , so we obtain (2.15).  $\square$

**2.3. Paths with jumps.** Let us explain how our construction of  $\mathbb{T}$  can be extended to càdlàg paths  $\omega$  (paths which are right-continuous and have left limits), see Figure 3; we apply the classical idea of embedding these paths into continuous paths by opening temporal windows at times of jumps and considering interpolated continuous paths (this idea has been used for the rough paths theory in [35]).

Let  $\mathbb{G}$  be the set of points  $(t, x)$  such that  $0 \leq t \leq 1$  and  $x$  is between  $\omega(t-)$  and  $\omega(t)$ . This is the graph of  $\omega$  augmented by the segments joining  $(t, \omega(t-))$  and  $(t, \omega(t))$ . Then define

$$\delta((t, x), (t, x')) := |x' - x|$$

and

$$\delta((s, x), (t, x')) := x + x' - 2 \left( \inf_{(s, t)} \omega \wedge x \wedge x' \right)$$

if  $s < t$ . If  $\omega$  is continuous, then  $\mathbb{G}$  and  $[0, 1]$  are naturally identified, in such a way that  $\delta$  coincides with the previous definition (2.1).

Let us say that two points of  $\mathbb{G}$  satisfy  $(t, x) \leq (t', x')$  if either  $t < t'$ , or  $t = t'$  and  $x$  is between  $\omega(t-)$  and  $x'$ . This is a total order, and  $\mathbb{G}$  can be endowed with the topology generated by open intervals for this order; actually, this topology coincides with the topology of  $\mathbb{G}$  considered as a subset of  $\mathbb{R}^2$ .

**PROPOSITION 2.4.** *The map  $\delta$  is a semi-distance on  $\mathbb{G}$ , and  $\mathbb{T} = (\mathbb{G}/\delta, \delta)$  is a compact real tree. Actually, there exists a continuous map  $\omega'$  such that  $\omega$  is obtained from  $\omega'$  by an increasing (not necessarily surjective) time change, and  $\mathbb{T}$  is the tree associated to  $\omega'$ .*

**PROOF.** Suppose that  $\omega$  is not continuous (the result is evident otherwise). Let  $J$  be the set of times where  $\omega$  jumps, and let  $(S(t); t \in J)$  be a family of (strictly) positive numbers such that  $\sum S(t) = 1$ . Let

$$\Lambda(t, x) := \frac{1}{2} \left( t + \sum_{u < t} S(u) + S(t) \frac{x - \omega(t-)}{\omega(t) - \omega(t-)} 1_J(t) \right).$$

Then  $\Lambda$  is an increasing bijection from  $\mathbb{G}$  onto  $[0, 1]$ , so  $\mathbb{G}$  and  $[0, 1]$  can be identified, and previous results on the tree representation for continuous functions defined on  $[0, 1]$  can also be applied to continuous functions on  $\mathbb{G}$ . Thus, in order to prove the proposition, it is sufficient to find a map  $\omega'$  defined on  $\mathbb{G}$ . Put  $\omega'(t, x) := x$ . It induces the semi-distance

$$\omega'(s, x) + \omega'(t, x') - 2 \inf_{[(s, x), (t, x')]} \omega' = \delta((s, x), (t, x')),$$

so its tree is  $\mathbb{T}$ . Moreover,  $\omega = \omega' \circ Q$  for the increasing time change  $Q(t) := (t, \omega(t))$ .  $\square$

In this setting, let  $\pi$  be the projection of  $\mathbb{G}$  on  $\mathbb{T}$ . We extend the notation (2.13) by

$$O := \pi(U, \omega(U) \wedge \omega(U-)), \quad A := \pi(0, \omega(0)), \quad B := \pi(1, \omega(1)),$$

where  $U$  is a time at which  $\omega(U) \wedge \omega(U-) = \inf \omega$ . Let  $E^{\nearrow}$  be the set of  $(t, x)$  in  $\mathbb{G}$  such that  $\omega(s) > x$  for any  $t < s < t + \varepsilon$  and some  $\varepsilon > 0$ , define  $E^{\nwarrow}$  similarly, and let

$$\omega^{\nearrow} := \int \sum_{s: (s, x) \in E^{\nearrow}} \delta_s dx, \quad \omega^{\nwarrow} := \int \sum_{s: (s, x) \in E^{\nwarrow}} \delta_s dx.$$

Notice also that all the points of  $\pi^{-1}(\tau)$  are at the same level; we let  $\tau^{\nearrow}$  and  $\tau^{\nwarrow}$  be the infimum and supremum of the time component of this set.

PROPOSITION 2.5. *The measures  $\omega^{\nearrow}$  and  $\omega^{\nwarrow}$  are the images of  $\lambda$  by  $\tau \mapsto \tau^{\nearrow}$  and  $\tau \mapsto \tau^{\nwarrow}$  [after restricting  $\lambda$  as in Proposition 2.2 if (2.2) does not hold]. The statements of Proposition 2.3 about the finite variation case again hold true.*

PROOF. Let us use the notation of the proof of Proposition 2.4. The set  $E^{\nearrow}$  is the set of beginnings of excursions of  $\omega'$ , so  $\omega^{\nearrow}$  is the projection on the time component of  $(\omega')^{\nearrow}$ ; we deduce the first statement. Moreover,  $\omega'$  is monotone on the intervals corresponding to the jumps of  $Q$ , so the total variations of  $\omega$  and  $\omega'$  coincide (a more general result will be proved in Theorem 3.10), and the Lebesgue–Stieltjes measure of  $\omega$  is again deduced from its analogue for  $\omega'$  by projection on the time component.  $\square$

**3.  $p$ -variation and trees.** Let us now assume that  $\omega$  has finite  $p$ -variation for some  $p \geq 1$ , so that

$$(3.1) \quad V_p(\omega) := \sup_{(t_i)} V_p(\omega, (t_i)) := \sup_{(t_i)} \sum_i |\omega(t_{i+1}) - \omega(t_i)|^p < \infty,$$

where the supremum is with respect to all the subdivisions of  $[0, 1]$  (notice that a nonconstant continuous map cannot have finite  $p$ -variation for  $p < 1$ ). Let us first assume that  $\omega$  is continuous (the càdlàg case will be dealt with in Section 3.3). We first want to describe the property (3.1) by means of the geometry of  $\mathbb{T}$ . In particular,  $V_p(\omega) < \infty$  implies  $V_q(\omega) < \infty$  for  $q \geq p$ , and we are interested in the variation index

$$(3.2) \quad \mathcal{V}(\omega) := \inf\{p \geq 1; V_p(\omega) < \infty\}.$$

3.1. *The variation index.* Let us recall that we have defined in (2.6) approximations  $\mathbb{T}^a$  of  $\mathbb{T}$  obtained by trimming the tree, that  $N^a$ , defined by (2.7), is the number of leaves of  $\mathbb{T}^a$ , and that the flattened path  $\omega^a$  of (2.9) is associated to  $\mathbb{T}^a$ ; let  $L^a := \lambda(\mathbb{T}^a)$  be its total length. As  $a \downarrow 0$ , each branch of  $\mathbb{T}^a$  grows at unit speed at its leaves, so

$$(3.3) \quad L^a = \int_a^\infty N^b db.$$

If  $\pi(0) = \pi(1)$ , we deduce from Proposition 2.3 that  $L^a$  is the mass of the positive part of  $d\omega^a$ , so, by applying (2.11) and (2.10),

$$\begin{aligned} L^a &= \omega(T_1^a) + \sum_{i=2}^{N^a} (\omega(T_i^a) - \omega(U_{i-1}^a)) \\ &= \omega(T_1^a) + \sum_{i=2}^{N^a} (\omega(T_i^a) - \omega(S_{i-1}^a) + a) \\ &= \sum_{i=1}^{N^a} (\omega(T_i^a) - \omega(S_{i-1}^a)) + (N^a - 1)a. \end{aligned}$$

If  $\pi(0) \neq \pi(1)$ , then this equation has to be corrected as in (2.15); notice, however, that the correction is bounded, so if  $\omega$  has infinite variation, then

$$(3.4) \quad L^a \sim \sum_{i=1}^{N^a-1} (\omega(T_i^a) - \omega(S_{i-1}^a)) + aN^a \quad \text{as } a \downarrow 0.$$

Thus  $L^a$  is easily estimated from the path  $\omega$ , the times  $S_i^a$  and  $T_i^a$ , and the number  $N^a$  of (2.7).

We consider two other metric characteristics of  $\mathbb{T}$ , namely its upper box (or Minkowski) dimension (see, for instance, [12]) defined by

$$\overline{\dim} \mathbb{T} := \limsup_{a \downarrow 0} \frac{\log \mathcal{N}(a)}{\log(1/a)}$$

where  $\mathcal{N}(a)$  is the minimal number of balls of radius  $a$  which are needed to cover  $\mathbb{T}$ , and the index

$$\mathcal{H}(\mathbb{T}) := \inf \left\{ p \geq 1; \int_{\mathbb{T}} (h(\tau))^{p-1} \lambda(d\tau) < \infty \right\}$$

where  $h(\tau)$  is the height of the highest branch above  $\tau$ . The aim of this subsection is to prove that all these quantities are related to the variation index  $\mathcal{V}(\omega)$  defined in (3.2), and in particular prove the result announced in (1.1).

**THEOREM 3.1.** *Let  $\omega$  be a (nonconstant) continuous function. Then*

$$\begin{aligned} \mathcal{V}(\omega) &= \mathcal{H}(\mathbb{T}) \\ &= \limsup_{a \rightarrow 0} \frac{\log L^a}{\log(1/a)} + 1 \\ &= \limsup_{a \rightarrow 0} \frac{\log N^a}{\log(1/a)} \vee 1 \\ &= \overline{\dim} \mathbb{T}. \end{aligned}$$

**PROOF.** Denoting by  $I_1, \dots, I_5$  the successive terms of the theorem, we prove that

$$I_1 \leq I_2 \leq I_3 \leq I_4 \leq I_5 \leq I_1.$$

These five inequalities are proved in the five following steps.

**PROOF OF  $I_1 \leq I_2$ .** Let  $s < t$  be two times, and let  $\tau_0$  be the most recent common ancestor of  $\pi(s)$  and  $\pi(t)$ . Then

$$\ell(\tau_0) = \min_{[s,t]} \omega$$

so

$$|\omega(t) - \omega(s)| \leq \max(\omega(s) - \ell(\tau_0), \omega(t) - \ell(\tau_0))$$

and

$$|\omega(t) - \omega(s)|^p \leq (\omega(s) - \ell(\tau_0))^p + (\omega(t) - \ell(\tau_0))^p.$$

On the other hand,

$$\begin{aligned} (\omega(t) - \ell(\tau_0))^p &= p \int_{[\tau_0, \pi(t)]} (\omega(t) - \ell(\tau))^{p-1} \lambda(d\tau) \\ &\leq p \int_{[\tau_0, \pi(t)]} (h(\tau) - h(\pi(t)))^{p-1} \lambda(d\tau) \\ &\leq p \int_{[\tau_0, \pi(t)]} (h(\tau))^{p-1} \lambda(d\tau) \end{aligned}$$

where we have used in the second line the property

$$\begin{aligned} h(\tau) - h(\pi(t)) &= \max_{[\tau^{\nearrow}, \tau^{\searrow}]} \ell - \ell(\tau) - \max_{[\pi(t)^{\nearrow}, \pi(t)^{\searrow}]} \ell + \omega(t) \\ &\geq \omega(t) - \ell(\tau) \end{aligned}$$

valid for  $\pi(t)$  above  $\tau$ . The same property holds at time  $s$ , so by addition,

$$|\omega(t) - \omega(s)|^p \leq p \int_{[\pi(s), \pi(t)]} (h(\tau))^{p-1} \lambda(d\tau).$$

If  $(t_i)$  is a subdivision of  $[0, 1]$ , we can sum up these estimates for  $s = t_i$  and  $t = t_{i+1}$ . Since almost any  $\tau$  appears at most twice in the right-hand sides (at times  $\tau^{\nearrow}$  and  $\tau^{\searrow}$ ), we deduce

$$(3.5) \quad V_p(\omega) \leq 2p \int_{\mathbb{T}} (h(\tau))^{p-1} \lambda(d\tau).$$

In particular  $I_1 \leq I_2$ .

PROOF OF  $I_2 \leq I_3$ . It follows from  $\lambda = \lambda_2$  (Proposition 2.1) and from (3.3) that for  $p > 1$ ,

$$\int_{\mathbb{T}} (h(\tau))^{p-1} \lambda(d\tau) = \int_0^\infty a^{p-1} N^a da = (p-1) \int_0^\infty a^{p-2} L^a da.$$

We deduce that if  $L^a \leq Ca^{1-\kappa}$  for some  $\kappa < p$ , then the integral is finite, so  $I_2 \leq I_3$ .

PROOF OF  $I_3 \leq I_4$ . This inequality follows from (3.3).

PROOF OF  $I_4 \leq I_5$ . Above each  $\tau \in \partial \mathbb{T}^a$  there is a  $\tau'$  such that  $\delta(\tau, \tau') = a$ , and the  $N^a$  balls with centers  $\tau'$  and radius  $a$  are disjoint; this implies that the number of balls of radius  $a/2$  which is needed to cover  $\mathbb{T}$  is at least  $N^a$ ; we also have  $\overline{\dim} \mathbb{T} \geq 1$ , so we deduce that  $I_4 \leq I_5$ .



PROOF OF  $I_5 \leq I_1$ . For  $a > 0$ , let  $t_0 = 0$  and

$$t_{i+1} = \inf\{t \geq t_i; |\omega(t) - \omega(t_i)| \geq a\}.$$

Let  $\tau_i$  be the most recent common ancestor of  $\pi(t_i)$  and  $\pi(t_{i+1})$ , so that  $\ell(\tau_i) = \inf_{[t_i, t_{i+1}]} \omega$ . Consider the closed ball  $B_i$  of  $\mathbb{T}$  with center  $\tau_i$  and with radius  $2a$ , so that  $\pi([t_i, t_{i+1}])$  is included in this ball. Then the union of  $B_i$  is a covering of  $\mathbb{T}$ . Moreover, the number of these balls is dominated by  $V_p(\omega)/a^p$ , so the upper box dimension of  $\mathbb{T}$  is dominated by  $p$  as soon as  $p > \mathcal{V}(\omega)$ . We deduce that  $I_5 \leq I_1$ .  $\square$

REMARK 3.2. If  $\pi(0) = \pi(1)$ , we have

$$V_p(\omega) \geq \sum_{\tau \in \partial \mathbb{T}^a} \left( \left( \sup_{[\tau^{\nearrow}, \tau^{\searrow}]} \omega - \omega(\tau^{\nearrow}) \right)^p + \left( \sup_{[\tau^{\nearrow}, \tau^{\searrow}]} \omega - \omega(\tau^{\searrow}) \right)^p \right) = 2a^p N^a.$$

If  $\pi(0) \neq \pi(1)$ , we have to omit the first term for the first leaf of  $\mathbb{T}^a$  ( $\tau^{\nearrow}$  may be before time 0), and the second term for the last leaf of  $\mathbb{T}^a$  ( $\tau^{\searrow}$  may be after time 1). Thus

$$(3.6) \quad V_p(\omega) \geq a^p N^a$$

and the right-hand side can be doubled if  $\pi(0) = \pi(1)$ .

REMARK 3.3. Other related estimates of  $V_p(\omega)$  using numbers of up-crossings were previously known; see [4, 34].

REMARK 3.4. The link between the dimension of  $\mathbb{T}$  and the behavior of  $N^a$  is similar to the link between the dimension of the boundary of discrete trees and their growth (see page 201 of [29]).

REMARK 3.5. A more classical fractal dimension related to a path  $\omega$  is the dimension of its graph as a subset of  $\mathbb{R}^2$ . This dimension (which is bounded by 2) is of course generally different from the dimension of  $\mathbb{T}$ .

Other well-known notions of dimensions ([12]) are the packing dimension  $\dim_P \mathbb{T}$  and the Hausdorff dimension  $\dim_H \mathbb{T}$ , and we always have

$$(3.7) \quad \dim_H \mathbb{T} \leq \dim_P \mathbb{T} \leq \overline{\dim} \mathbb{T}.$$

Some of these inequalities may be strict. For instance, consider the path  $\omega$  which is affine on each interval  $[1/(n+1), 1/n]$ , and such that

$$\omega(1/(2k+1)) = 0, \quad \omega(1/(2k)) = 1/k^\alpha.$$

Then

$$V_p(\omega) = 2 \sum k^{-\alpha p}$$

for  $p \geq 1$ , so  $\overline{\dim} \mathbb{T} = \mathcal{V}(\omega) = 1/\alpha \vee 1$ . On the other hand, the tree is a star with a countable number of branches, and its Hausdorff and packing dimensions are 1.

On the other hand, if  $\omega$  has the same variation index  $\mathcal{V}(\omega)$  on any interval  $[s, t]$  with  $s < t$ , then any open subset of  $\mathbb{T}$  has the same upper box dimension, so in this case ([12])

$$\dim_P \mathbb{T} = \overline{\dim} \mathbb{T} = \mathcal{V}(\omega).$$

We will now see an example where the Hausdorff dimension is also equal to  $\mathcal{V}(\omega)$ .

**3.2. The fractional Brownian case.** We now consider the case where  $\omega$  is a typical path of a fractional Brownian motion  $W$ . This is a centered Gaussian process  $(W_t; t \in \mathbb{R})$  with covariance function

$$\text{cov}(W_s, W_t) = \frac{\sigma^2}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H})$$

for the Hurst parameter  $0 < H < 1$  and the coefficient  $\sigma^2 > 0$ . It satisfies the scaling property

$$(3.8) \quad (W_{ct}; t \in \mathbb{R}) \stackrel{\text{law}}{=} (c^H W_t; t \in \mathbb{R})$$

for  $c > 0$ . In this subsection, we let  $\omega$  be a path of  $W$  restricted to  $[0, 1]$  and extended to  $[-1, 2]$  by the technique of (2.12); we compute the Hausdorff dimension of the tree  $\mathbb{T}$ , and describe an estimator of  $H$  based on  $\mathbb{T}$ .

The property  $V_p(W) < \infty$  for  $p > 1/H$  is well known; it is classically obtained from the  $(1/p)$ -Hölder continuity of the paths, which itself is obtained by means of the Kolmogorov criterion and the estimation

$$\|W_t - W_s\|_q = C_q \sigma (t - s)^H$$

on the  $L^q$  norm of the increments for any  $q \geq 1$ . It actually follows from this estimation that the moments of  $V_p(W)$  are finite.

**PROPOSITION 3.6.** *For almost any path  $\omega$  of  $W$ , one has*

$$\dim_H \mathbb{T} = \mathcal{V}(\omega) = 1/H.$$

**PROOF.** The property  $\mathcal{V}(W) \leq 1/H$  follows from the discussion preceding the proposition. From (3.7) and Theorem 3.1, it is therefore sufficient to prove that  $\dim_H \mathbb{T} \geq 1/H$ . The constants involved in this proof depend on  $H$  and  $\sigma$ . It is known from [24] that

$$(3.9) \quad \mathbb{P} \left[ \inf_{[0, 1/2]} W > -u \right] = O(u^\gamma)$$

as  $u \downarrow 0$ , for any  $\gamma < 1/H - 1$ . Moreover, if  $(\mathcal{F}_t; 0 \leq t \leq 1)$  is the filtration of  $W$ , the conditional law of  $W_1 - W_{1/2}$  given  $\mathcal{F}_{1/2}$  is a Gaussian law with deterministic positive variance, so

$$(3.10) \quad \mathbb{P}[|W_1| < u \mid \mathcal{F}_{1/2}] \leq Cu.$$

The event  $\{\delta(0, 1) < u\}$  is included in the intersection of the two events of (3.9) and (3.10), so

$$\mathbb{P}[\delta(0, 1) < u] = O(u^{\gamma+1}).$$

We deduce that  $\delta(0, 1)^{-p}$  is integrable for  $p < 1/H$ . From the scaling property (3.8),  $\delta(s, t)^{-p}$  is also integrable, and

$$\mathbb{E}\delta(s, t)^{-p} = C(t - s)^{-pH},$$

so

$$(3.11) \quad \mathbb{E} \int_{\mathbb{T} \times \mathbb{T}} \delta(\tau_1, \tau_2)^{-p} \nu(d\tau_1) \nu(d\tau_2) = \mathbb{E} \int_0^1 \int_0^1 \delta(s, t)^{-p} ds dt < \infty$$

for the projection  $\nu$  of the Lebesgue measure of  $[0, 1]$  on  $\mathbb{T}$ . The double integral of the left-hand side is the  $p$ -energy of the measure  $\nu$  on the metric space  $(\mathbb{T}, \delta)$ . Its almost sure finiteness implies that  $\dim_H \mathbb{T} \geq p$  for any  $p < 1/H$  (see, for instance, [12]), so  $\dim_H \mathbb{T} \geq 1/H$ .  $\square$

Dimensions of Lévy trees have been computed in [8]. This includes our tree  $\mathbb{T}$  for  $H = 1/2$ , and for this tree, the exact Hausdorff measure has been obtained in [9]. Here, we do not look for a so precise result, but verify that the normalization of the length measure  $\lambda$  on  $\mathbb{T}^a$  converges to the measure  $\nu$  of the previous proof; the same property is verified for the uniform measure on leaves of  $\mathbb{T}^a$ . In this sense,  $\nu$  can be viewed as a uniform measure on the leaves of the tree. This will be a corollary of the following result (Proposition 3.8).

**PROPOSITION 3.7.** *For almost any path  $\omega$  of  $W$ , we have*

$$N^a \sim C(H) \sigma^{1/H} a^{-1/H}, \quad L^a \sim C(H) \frac{H}{1-H} \sigma^{1/H} a^{1-1/H}$$

as  $a \downarrow 0$ , for some  $C(H) > 0$ .

**PROOF.** Since  $N^a$  and  $L^a$  are related to each other by means of (3.3), it is sufficient to study  $N^a$ . Moreover,  $\sigma$  acts as a multiplicative coefficient on the path, so  $N^a$  for the process with parameter  $\sigma$  has the same law as  $N^{a/\sigma}$  for the process with parameter 1; thus it is sufficient to consider the case

$\sigma = 1$ . If  $p > 1/H$ , it follows from the finiteness of the moments of  $V_p(W)$  and from (3.6) that

$$(3.12) \quad \|N^a\|_q \leq Ca^{-p}$$

for any  $q \geq 1$  and some  $C = C(p, q, H)$ . In the two following steps, we study successively the expectation and the variance of  $N^a$ .

STUDY OF  $\mathbb{E}[N^a]$ . Consider in this proof the whole path  $(\omega(t); t \in \mathbb{R})$  of  $W$ , and its associated (noncompact) tree  $\mathbb{T}_{-\infty, +\infty}$ . For  $s < t$ , let  $N_{s,t}^a$ , respectively  $\tilde{N}_{s,t}^a$ , be the numbers of leaves of the trimmed tree  $\mathbb{T}_{-\infty, +\infty}^a$  such that  $s < \tau^\nearrow < \tau^\searrow < t$ , respectively  $s \leq \tau^\nearrow < t$ . Then

$$(3.13) \quad \tilde{N}_{s,t}^a - N_{s,t}^a \in \{0, 1\}, \quad N^a - N_{0,1}^a \in \{0, 1, 2\}$$

(actually one may have  $\tilde{N}_{s,t}^a - N_{s,t}^a = 2$  if  $s$  is some  $\tau^\nearrow$ , but this happens with zero probability for any fixed  $s$ ). On the other hand, it follows from the scaling property (3.8) of  $W$  that

$$\mathbb{E}[\tilde{N}_{0,1}^a] = \mathbb{E}[\tilde{N}_{0,a^{-1/H}}^1].$$

The law of  $W$  is shift invariant and  $[s, t) \mapsto \tilde{N}_{s,t}^1$  is additive, so  $\mathbb{E}[\tilde{N}_{s,t}^1]$  is proportional to  $t - s$ , and

$$\mathbb{E}[\tilde{N}_{0,1}^a] = a^{-1/H} \mathbb{E}[\tilde{N}_{0,1}^1].$$

Thus the result of the proposition holds in expectation for  $C(H) = \mathbb{E}[\tilde{N}_{0,1}^1]$ .

STUDY OF  $\text{var}(N^a)$ . It follows from (3.13) and the additivity of  $[s, t) \mapsto \tilde{N}_{s,t}^a$  that

$$|N_{s,u}^a + N_{u,t}^a - N_{s,t}^a| \leq 2$$

for  $s \leq u \leq t$ . Thus, by considering a regular subdivision of  $[0, 1]$  with mesh  $\Delta t$ , we have

$$(3.14) \quad \left| N^a - \sum N_{t_i, t_{i+1}}^a \right| \leq 2\Delta t^{-1} + 2.$$

Moreover,

$$\begin{aligned} \text{var}(N_{t_i, t_{i+1}}^a) &= \text{var}(N_{0, \Delta t}^a) = \text{var}(N_{0,1}^{a\Delta t^{-H}}) \\ &\leq \mathbb{E}[(N_{0,1}^{a\Delta t^{-H}})^2] \leq \mathbb{E}[(N^{a\Delta t^{-H}})^2] \leq Ca^{-2p}(\Delta t)^{2pH} \end{aligned}$$

for  $p > 1/H$ , where we have used the scaling property in the second equality, and (3.12) in the last inequality. Since  $N_{t_i, t_{i+1}}^a$  depends only on the increments of  $\omega$  on  $[t_i, t_{i+1}]$ , we deduce from the result (A.1) of Appendix A.1 that

$$\begin{aligned} \text{var}\left(\sum N_{t_i, t_{i+1}}^a\right) &\leq Ca^{-2p}(\Delta t)^{2pH} \sum_{k, j \leq \Delta t^{-1}} \frac{1}{1 + |k - j|^{1-H}} \\ (3.15) \quad &\leq C'a^{-2p}(\Delta t)^{2pH-H-1}, \end{aligned}$$

so, by joining (3.14) and (3.15),

$$\text{var}(N^a) \leq C(a^{-2p}(\Delta t)^{2pH-H-1} + (\Delta t)^{-2}).$$

We choose  $\Delta t \sim a^\alpha$  for  $0 < \alpha < 1/H$ , so

$$\text{var}(N^a) \leq C(a^{-2\alpha} + a^{-2p+\alpha(2pH-H-1)}).$$

By choosing  $p$  and  $\alpha$  close enough to  $1/H$ , we have

$$\text{var}(N^a) \leq Ca^{2\varepsilon-2/H}$$

for some  $\varepsilon > 0$ .

CONCLUSION OF THE PROOF. The two previous steps show that  $a^{1/H}N^a$  converges in  $L^2$  to a constant, and that the rate of convergence is at most of order  $a^\varepsilon$ . From the Borel–Cantelli lemma, the convergence is almost sure on a sequence  $a_n = n^{-\beta}$  for  $\beta$  large enough. Since  $a \mapsto N^a$  is monotone, we deduce from

$$a_{n+1}^{1/H}N^{a_n} \leq a^{1/H}N^a \leq a_n^{1/H}N^{a_{n+1}}$$

for  $a_{n+1} \leq a \leq a_n$ , that the convergence is actually almost sure as  $a \downarrow 0$ .  $\square$

PROPOSITION 3.8. *For almost any path  $\omega$  of  $W$ , the measures*

$$\nu_1^a := \frac{1}{N^a} \sum_{\tau \in \partial \mathbb{T}^a} \delta_\tau \quad \text{and} \quad \nu_2^a := \frac{1}{L^a} \lambda|_{\mathbb{T}^a}$$

*converge weakly to the projection  $\nu$  on  $\mathbb{T}$  of the Lebesgue measure of  $[0, 1]$ .*

PROOF. Let  $\mu_1^a$  and  $\mu_2^a$  be the images of  $\nu_1^a$  and  $\nu_2^a$  by  $\tau \mapsto \tau^\nearrow$ . One has  $\pi(\tau^\nearrow) = \tau$ , so  $\nu_1^a$  and  $\nu_2^a$  are the images of  $\mu_1^a$  and  $\mu_2^a$  by  $\pi$ . Since  $\pi$  is continuous, it is sufficient to prove that  $\mu_1^a$  and  $\mu_2^a$  converge weakly to the Lebesgue measure of  $[0, 1]$ , and therefore that  $\mu_1^a([s, t])$  and  $\mu_2^a([s, t])$  converge to  $t - s$ . But  $\mu_1^a([s, t])$  counts the proportion of leaves of  $\mathbb{T}^a$  which satisfy  $s \leq \tau^\nearrow \leq t$ ; the number of such leaves is close to the number  $\tilde{N}_{s,t}^a$  of the proof of Proposition 3.7; it can be estimated from Proposition 3.7 and the scaling property, and we can conclude. The study of  $\mu_2^a$  is similar.  $\square$

We can deduce estimators for  $H$  from Proposition 3.7. Our result is an alternative to the generalized quadratic variation approach [16]. For instance, we can consider  $N^{2a}/N^a$  or  $L^{2a}/L^a$ , so that the unknown coefficient  $\sigma$  is eliminated. However, we can also use

$$\lim_{a \downarrow 0} \frac{aN^a}{L^a} = \frac{1}{H} - 1.$$

Roughly speaking, the estimator  $aN^a/L^a$  counts the normalized number of changes in the sense of variation of  $\omega^a$ . The smaller  $H$  is, the more often the sense of variation of  $\omega^a$  changes. From (3.4), we deduce the following result.

PROPOSITION 3.9. *The Hurst parameter  $H$  of the fractional Brownian motion  $(W_t; 0 \leq t \leq 1)$  can be estimated from the relation*

$$\lim_{a \downarrow 0} \frac{1}{aN^a} \sum_{i=1}^{N^a-1} (W_{T_i^a} - W_{S_{i-1}^a}) = \frac{2H-1}{1-H}$$

which holds almost surely, where  $N^a$ ,  $S_i^a$ ,  $T_i^a$  were defined in (2.7).

3.3. *The case with jumps.* We now consider a càdlàg path  $\omega$ . We have seen in Proposition 2.4 how it can be written as a time-changed path  $\omega = \omega' \circ Q$  for a continuous  $\omega'$  defined on  $\mathbb{G}$ , and the trees of  $\omega$  and  $\omega'$  coincide. Actually, the variations also coincide, so the tree  $\mathbb{T}$  can again be used to study the variations of  $\omega$ .

THEOREM 3.10. *Let  $\omega$  be a càdlàg path and  $\omega'$  the associated continuous path. One has  $V_p(\omega) = V_p(\omega')$  for any  $p \geq 1$ . In particular,  $\mathcal{V}(\omega') = \mathcal{V}(\omega)$  and Theorem 3.1 again holds.*

PROOF. The relation  $\omega = \omega' \circ Q$  immediately implies  $V_p(\omega) \leq V_p(\omega')$ . In order to verify the reverse inequality, we notice that when computing  $V_p(\omega')$ , it is sufficient to consider subdivisions  $(t_i)$  consisting of local extrema of  $\omega'$ ; thus these times are in the closure of the image of  $Q$ ; consequently, from the continuity of  $\omega'$ , it is sufficient to consider times in the image of  $Q$ , so that we can conclude.  $\square$

We now give applications of the tree representation to martingales and Lévy processes. In the following result, we recover with our method a result of [31] (which was given in discrete time). Notice, however, that our results are only for the real-valued case, whereas [31] considers the Banach space-valued case.

PROPOSITION 3.11. *Consider a purely discontinuous martingale  $X = (X_t; 0 \leq t \leq 1)$  for a filtration  $(\mathcal{F}_t; 0 \leq t \leq 1)$ . Let  $1 < p < 2$ ; then*

$$(3.16) \quad \mathbb{E}[V_p(X)] \leq C_p \mathbb{E} \sum |\Delta X_t|^p.$$

PROOF. The proof is divided into two steps; in the first step, we reduce the problem to a particular case.

STEP 1. Let  $S_0 := 0$  and  $(S_k; k \geq 1)$  be the times of jumps of an independent standard Poisson process, and consider

$$X_t^\varepsilon = \sum X_{\varepsilon S_k} 1_{\{\varepsilon S_k \leq t < \varepsilon S_{k+1}\}}$$

( $X$  is supposed to be constant after time 1). Then  $X^\varepsilon$  is a martingale in its filtration; if the proposition were proved for  $X^\varepsilon$ , we would have

$$\begin{aligned}
 V_p(X^\varepsilon) &\leq C_p \mathbb{E} \sum_k |X_{\varepsilon S_{k+1}} - X_{\varepsilon S_k}|^p \\
 (3.17) \quad &\leq C'_p \mathbb{E} \sum_k \left( \sum |\Delta X_t|^2 1_{\{\varepsilon S_k \leq t < \varepsilon S_{k+1}\}} \right)^{p/2} \\
 &\leq C'_p \mathbb{E} \sum |\Delta X_t|^p
 \end{aligned}$$

where we have used in the second line the classical Burkholder–Davis–Gundy inequalities; it is then sufficient to let  $\varepsilon$  tend to 0. Thus it is sufficient to prove the result for martingales varying only on a sequence of totally inaccessible stopping times. By separating the positive and negative parts of the jumps, such a martingale is the difference of two martingales with finite variation and with no negative jump, so we only have to prove the result for these martingales.

STEP 2. We suppose therefore that  $X$  has finite variation with positive jumps at a sequence of stopping times  $S_k$ . Thus the positive part  $dX^+ = X^\nearrow$  of the Lebesgue–Stieltjes measure of  $X$  is purely atomic; it is carried by the times of jumps of  $X$ . Let  $\tau$  be in  $\mathbb{T}$ ; it is the projection of some  $(\tau^\nearrow, x)$  of  $\mathbb{G}$ , and

$$h(\tau) = \sup\{X_s - x; \tau^\nearrow \leq s \leq T(\tau^\nearrow, x)\}$$

with

$$T(t, x) := \inf\{s \geq t; X_s \leq x\}.$$

Then (3.5) implies that

$$\begin{aligned}
 V_p(X) &\leq 2p \int_{\inf X}^{X_0} \sup\{X_s - x; s \leq T(0, x)\}^{p-1} dx \\
 &\quad + 2p \sum_{t \in J} \int_{X_{t-}}^{X_t} \sup\{X_s - x; t \leq s \leq T(t, x)\}^{p-1} dx
 \end{aligned}$$

where  $J = \{S_k; k \geq 1\}$ . The first term corresponds to the integral on the arc  $[A, O]$  of  $\mathbb{T}$ , on which  $\tau^\nearrow \leq 0$ ; its expectation is dominated by the expectation of  $|X_1 - X_0|^p$  (Doob's inequality) which can be estimated by the right-hand side of (3.16) with the technique of (3.17). The second term corresponds to the integral on the remaining part of the tree, for which  $\tau^\nearrow \in J$ . In order to estimate it, consider some jump  $S = S_k$  and notice that since  $X$  is a martingale with no negative jump,

$$\mathbb{P}[\sup\{X_s - x; S \leq s \leq T(S, x)\} \geq a \mid \mathcal{F}_S] \leq \frac{X_S - x}{a}$$

for  $X_{S-} \leq x \leq X_S$  and  $a \geq X_S - x$ . We deduce that

$$\mathbb{E}[\sup\{X_s - x; S \leq s \leq T(S, x)\}^{p-1} \mid \mathcal{F}_S] \leq (X_S - x) \int_{X_S - x}^M a^{p-3} da,$$

so

$$\mathbb{E}\left[\int_{X_{S-}}^{X_S} \sup\{X_s - x; S \leq s \leq T(t, x)\}^{p-1} dx \mid \mathcal{F}_S\right] \leq (\Delta X_S)^p / (p(2-p))$$

and we can conclude by summing on the times of jumps  $S = S_k$ .  $\square$

We now give for Lévy processes the analogue of Proposition 3.6.

**PROPOSITION 3.12.** *Let  $X$  be an  $\alpha$ -stable Lévy process. Then, for almost any path  $\omega$  of  $X$ ,*

$$\dim_H \mathbb{T} = \overline{\dim} \mathbb{T} = \mathcal{V}(\omega) = \alpha \vee 1.$$

**PROOF.** For  $\alpha < 1$ , the process has finite variation, so  $\mathcal{V}(X) = 1$  and the dimension is 1. For  $\alpha \geq 1$ , the fact that  $\mathcal{V}(X) \leq \alpha$  is classical and can be deduced from Proposition 3.11; thus

$$1 \leq \dim_H \mathbb{T} \leq \overline{\dim} \mathbb{T} = \mathcal{V}(X) \leq \alpha.$$

Our result is therefore proved for  $\alpha = 1$ . Suppose now  $\alpha > 1$ . We will use the notation

$$\delta(s, t) = \delta((s, X_s), (t, X_t)) = X_s + X_t - 2 \inf_{[s, t]} X.$$

It is known (Proposition VIII.2 of [3]) that

$$\mathbb{P}\left[\inf_{[0, 1/2]} X > -u\right] \leq Cu^{\alpha\beta} = O(u^{\alpha-1})$$

as  $u \downarrow 0$ , for  $\beta = \mathbb{P}[X_t \leq 0] \geq (\alpha - 1)/\alpha$ . We also have

$$\begin{aligned} \mathbb{P}[|X_1| < u \mid \mathcal{F}_{1/2}] &\leq \sup_x \mathbb{P}[x - u < X_1 - X_{1/2} < x + u] \\ &= \sup_x \mathbb{P}[x - u < X_{1/2} < x + u] = O(u) \end{aligned}$$

because  $X_{1/2}$  has a bounded density, so by taking the intersection of these two events,

$$\mathbb{P}[\delta(0, 1) < u] = O(u^\alpha).$$

We deduce that  $\delta(0, 1)^{-p}$  is integrable for any  $p < \alpha$ . The variables  $\delta(s, t)$  satisfy the same property, and by scaling,

$$\mathbb{E}\delta(s, t)^{-p} = C(t - s)^{-p/\alpha}.$$

This can be used to prove (3.11) for any  $p < \alpha$ , so we deduce as in Proposition 3.6 that the Hausdorff dimension is bounded below by  $\alpha$ .  $\square$



REMARK 3.13. Another real tree, called the Lévy tree, has been associated to  $X$  in [19] when  $X$  has only positive jumps. This tree is different from  $\mathbb{T}$  but is related to it; times which project on the same point of  $\mathbb{T}$  also project on the same point of the Lévy tree, but an arc of  $\mathbb{T}$  associated to a jump of  $X$  is concentrated in the Lévy tree into a single point.

Let us now give an analogue of Proposition 3.7 for Lévy processes.

PROPOSITION 3.14. *Let  $X$  be a Lévy process. Suppose that almost surely,  $X$  has no interval on which it is monotone, and define*

$$\xi(a) = \mathbb{E}[S^a + T^a]$$

for

$$T^a := \inf \left\{ t; X_t < \sup_{[0,t]} X - a \right\}, \quad S^a := \inf \left\{ t; X_t > \inf_{[0,t]} X + a \right\}.$$

Then  $\lim_0 \xi = 0$ , and  $\xi(a)N^a(X)$  (for the process  $X$  on the time interval  $[0, 1]$ ) converges in probability to 1 as  $a \downarrow 0$ . If  $\xi(a) = O(a^\alpha)$  for some  $\alpha > 0$ , then the convergence is almost sure.

When the assumption about  $X$  is not satisfied, then  $X$  or  $-X$  is the sum of a subordinator and a compound Poisson process. In this case,  $\mathbb{T}$  is finite, so  $N^a$  is bounded.

PROOF OF PROPOSITION 3.14. Consider the times  $T_i^a = T_i^a(X)$  and  $S_i^a = S_i^a(X)$  defined by (2.7). On the other hand, notice that our assumption implies that  $S^a$  and  $T^a$  tend almost surely to 0 as  $a \downarrow 0$ . Since  $X$  is a Lévy process, times  $T_{i+1}^a - S_i^a$  and  $S_i^a - T_i^a$  are independent, and have the same law as  $T^a$  and  $S^a$ . Thus

$$\sup_{0 \leq t \leq k\mu} \left( X_t - \inf_{[0,t]} X \right) \geq \sup_{1 \leq j \leq k} \left( \sup_{(j-1)\mu \leq t \leq j\mu} \left( X_t - \inf_{[(j-1)\mu, t]} X \right) \right)$$

and the right-hand side is the supremum of  $k$  independent identically distributed variables, so

$$\mathbb{P}[S^a > k\mu] = \mathbb{P} \left[ \sup_{0 \leq t \leq k\mu} \left( X_t - \inf_{[0,t]} X \right) < a \right] \leq (\mathbb{P}[S^a \geq \mu])^k$$

for  $\mu > 0$ . This probability is smaller than 1 from our assumption on  $X$ . We deduce that the moments of  $S^a$  (and  $T^a$ ) are finite, so  $\lim_0 \xi = 0$  and

$$\mathbb{P}[S^a > 2k\mathbb{E}[S^a]] \leq 1/2^k.$$

Thus  $S^a/(2\mathbb{E}[S^a])$ , and similarly  $T^a/(2\mathbb{E}[T^a])$ , are dominated by a geometric variable, so the variances of  $S^a$  and  $T^a$  are dominated by  $(\mathbb{E}[S^a])^2$  and  $(\mathbb{E}[T^a])^2$ . Thus

$$(3.18) \quad \mathbb{E}[S_n^a] = n\xi(a), \quad \text{var}(S_n^a) = n(\text{var}(S^a) + \text{var}(T^a)) \leq Cn\xi(a)^2.$$

If  $n = n(a) \uparrow \infty$  as  $a \downarrow 0$ , then  $n(a)^{-1}\xi(a)^{-1}S_{n(a)}$  has expectation 1 and has a variance dominated by  $1/n(a)$ ; in particular it converges in probability to 1. By taking  $n = n(a, \pm) \sim (1 \pm \varepsilon)\xi(a)^{-1}$ , we see from (3.18) and the definition of  $N^a$  in (2.7) that  $N^a$  is between  $n(a, -)$  and  $n(a, +)$  with a high probability, so the convergence in probability of the proposition is proved. Moreover, for the second statement, it follows from the Borel–Cantelli lemma that  $n(a_k)^{-1}\xi(a_k)^{-1}S_{n(a_k)}$  converges almost surely to 1 as soon as  $\sum 1/n(a_k) < \infty$ . We can apply this result to the above  $n = n(a_k, \pm)$  for  $a_k = 1/k^\beta$  and  $\beta$  large enough, and we deduce that  $\xi(a_k)N^{a_k}$  converges almost surely to 1. We conclude as in Proposition 3.7 from the monotonicity of  $N^a$ .  $\square$

The almost sure convergence holds in particular for  $\alpha$ -stable processes such that  $|X|$  is not a subordinator. In this case indeed,  $\xi(a)$  is proportional to  $a^\alpha$  from the scaling property. For the standard Brownian motion,  $S^1$  and  $T^1$  are the first hitting time of 1 by a reflected Brownian motion, and have expectation 1. Thus  $\xi(a) = 2a^2$  and  $N^a \sim 1/(2a^2)$ . This means that  $C(1/2) = 1/2$  in Proposition 3.7.

We can deduce an estimation of  $L^a$  when the process has infinite variation. However, (3.4) cannot be directly applied; one has to use the associated continuous path, since times  $S_i^a$  and  $T_i^a$  can be jump times.

#### 4. Integrals and trees.

4.1. *An integral on the tree.* We now want to integrate some bounded function  $\rho(t)$  against  $\omega$ . First suppose that  $\omega$  is continuous and  $\pi(0) = \pi(1)$ . Let us remember (Proposition 2.3) that if  $\omega$  has finite variation, then  $\omega^{\nearrow}$  and  $\omega^{\searrow}$  are finite measures, and are the positive and negative parts of the Lebesgue–Stieltjes measure  $d\omega$ ; moreover, since the images of the finite length measure  $\lambda$  by  $\tau \mapsto \tau^{\nearrow}$  and  $\tau \mapsto \tau^{\searrow}$  are respectively  $\omega^{\nearrow}$  and  $\omega^{\searrow}$  (Proposition 2.2), we have

$$\int_0^1 \rho(t) \omega^{\nearrow}(dt) = \int_{\mathbb{T}} \rho(\tau^{\nearrow}) \lambda(d\tau), \quad \int_0^1 \rho(t) \omega^{\searrow}(dt) = \int_{\mathbb{T}} \rho(\tau^{\searrow}) \lambda(d\tau),$$

so

$$(4.1) \quad \int_0^1 \rho d\omega = \int_{\mathbb{T}} (\rho(\tau^{\nearrow}) - \rho(\tau^{\searrow})) \lambda(d\tau).$$

If  $\pi(0) \neq \pi(1)$ , we extend  $\omega$  to  $[-1, 2]$  as in (2.12), put  $\rho(t) = 0$  for  $t \notin [0, 1]$ , and we can use the same formula to define the integral; actually, with the notation (2.13), we have

$$(4.2) \quad \int_0^1 \rho d\omega = \int_{\mathbb{T} \setminus [A, B]} (\rho(\tau^{\nearrow}) - \rho(\tau^{\nwarrow})) \lambda(d\tau) - \int_{[A, O]} \rho(\tau^{\nwarrow}) \lambda(d\tau) + \int_{[O, B]} \rho(\tau^{\nearrow}) \lambda(d\tau).$$

In this form, one can notice that the integral on  $[0, 1]$  depends on  $\rho$  and  $\omega$  on  $[0, 1]$ , and not on the extension of  $\omega$  out of  $[0, 1]$ .

More generally, even if  $\omega$  has infinite variation, we can define the integral by the right-hand side of (4.1) or (4.2), provided

$$(4.3) \quad \int_{\mathbb{T}} |\rho(\tau^{\nearrow}) - \rho(\tau^{\nwarrow})| \lambda(d\tau) < \infty.$$

Notice that the right-hand side of (4.1) is the limit as  $a \downarrow 0$  of the integral on the trimmed tree  $\mathbb{T}^a$  which is the tree of  $\omega^a$  defined by (2.9), so  $\int \rho d\omega$  is the limit of  $\int \rho d\omega^a$ . This means that in this sense our approach is similar to other approaches using a regularization of  $\omega$ ; another example for which there has been a lot of work recently is the Russo–Vallois approach [33].

We now verify that we can apply our technique in the Young framework.

**THEOREM 4.1.** *Assume that  $\omega$  is continuous. One has*

$$(4.4) \quad \int_{\mathbb{T}} |\rho(\tau^{\nearrow}) - \rho(\tau^{\nwarrow})| \lambda(d\tau) \leq CV_p(\omega)^{1/p} (V_q(\rho)^{1/q} + \sup |\rho|)$$

for some  $C = C(p, q)$ , as soon as  $1/p + 1/q > 1$ . Thus (4.3) is satisfied as soon as  $1/\mathcal{V}(\omega) + 1/\mathcal{V}(\rho) > 1$ , and in this case we can define  $\int \rho d\omega$  by the right-hand side of (4.1) or (4.2). It satisfies

$$(4.5) \quad \left| \int_0^1 \rho d\omega \right| \leq CV_p(\omega)^{1/p} (V_q(\rho)^{1/q} + \sup |\rho|)$$

for  $1/p + 1/q > 1$ . Moreover, this integral coincides with the Riemann–Stieltjes integral constructed by Young [36] (see also [22, 23]); this means that

$$\int_0^1 \rho d\omega = \lim \sum_i \rho(s_i) (\omega(t_{i+1}) - \omega(t_i))$$

for  $t_i \leq s_i \leq t_{i+1}$ , as the mesh of the subdivision  $(t_i)$  of  $[0, 1]$  tends to 0. The integral  $\int_s^t \rho d\omega$  can be defined similarly by replacing  $\rho$  by  $\rho 1_{(s, t]}$ ; it satisfies the Chasles relation, and

$$(4.6) \quad V_p \left( \int_0^\cdot \rho d\omega \right)^{1/p} \leq CV_p(\omega)^{1/p} (V_q(\rho)^{1/q} + \sup |\rho|).$$

PROOF. Let us first assume  $\pi(0) = \pi(1)$ . It follows from the disintegration formula  $\lambda = \lambda_2$  of Proposition 2.1 that

$$I_a := -\frac{\partial}{\partial a} \int_{\mathbb{T}^a} |\rho(\tau^{\nearrow}) - \rho(\tau^{\searrow})| \lambda(d\tau) = \sum_{\tau \in \partial \mathbb{T}^a} |\rho(\tau^{\nearrow}) - \rho(\tau^{\searrow})|.$$

Define  $0 \leq r < 1$  by  $1/q + r/p = 1$ . Then

$$\begin{aligned} I_a &\leq \left( \sum_{\tau \in \partial \mathbb{T}^a} |\rho(\tau^{\nearrow}) - \rho(\tau^{\searrow})|^q \right)^{1/q} (N^a)^{r/p} \\ (4.7) \quad &\leq \frac{1}{2^{r/p} a^r} V_q(\rho)^{1/q} V_p(\omega)^{r/p} \end{aligned}$$

from Hölder's inequality and (3.6). Consequently,  $I_a$  is of order  $1/a^r$  and is integrable with respect to  $a$  near 0; more precisely, with  $\|\omega\| = \sup \omega - \inf \omega$ ,

$$\begin{aligned} \int_{\mathbb{T}} |\rho(\tau^{\nearrow}) - \rho(\tau^{\searrow})| \lambda(d\tau) &= \int_0^{\|\omega\|} I_a da \\ (4.8) \quad &\leq \frac{1}{2^{r/p}(1-r)} \|\omega\|^{1-r} V_q(\rho)^{1/q} V_p(\omega)^{r/p} \\ &\leq \frac{1}{2^{1/p}(1-r)} V_q(\rho)^{1/q} V_p(\omega)^{1/p} \end{aligned}$$

where we have used  $V_p(\omega) \geq 2\|\omega\|^p$  in the last line. If  $A = \pi(0) \neq \pi(1) = B$ , we decompose  $\mathbb{T}$  into  $[A, B]$  and  $\mathbb{T} \setminus [A, B]$ ; we can apply the above procedure to the integral on the latter part, and again prove (4.8), but without the factor 2. On the other hand,  $[A, B]$  has finite length so the integral is finite on it; more precisely,

$$\begin{aligned} \int_{[A, B]} |\rho(\tau^{\nearrow}) - \rho(\tau^{\searrow})| \lambda(d\tau) &= \int_{[A, O]} |\rho(\tau^{\searrow})| \lambda(d\tau) + \int_{[O, B]} |\rho(\tau^{\nearrow})| \lambda(d\tau) \\ &\leq \delta(0, 1) \sup |\rho| \leq 2V_p(\omega)^{1/p} \sup |\rho|. \end{aligned}$$

The result (4.4) follows by adding these two estimates. Thus we can define the integral  $\int_0^1 \rho d\omega$  by (4.1); this integral satisfies (4.5), and similarly,

$$\left| \int_s^t \rho d\omega \right| \leq C V_p(\omega; s, t)^{1/p} (V_q(\rho)^{1/q} + \sup |\rho|)$$

where the  $p$ -variation of  $\omega$  is limited to  $[s, t]$ . One easily deduces (4.6) by applying

$$(4.9) \quad \sum_i V_p(\omega; t_i, t_{i+1}) \leq V_p(\omega).$$

More precisely, by considering the variations of  $\rho$  and  $\omega$  on  $[s, t]$ ,

$$\begin{aligned}
& \left| \int_s^t \rho d\omega - \rho(s)(\omega(t) - \omega(s)) \right| \\
&= \left| \int_s^t (\rho(\cdot) - \rho(s)) d\omega \right| \\
&\leq C V_p(\omega; s, t)^{1/p} (V_q(\rho; s, t)^{1/q} + \sup |\rho(\cdot) - \rho(s)|) \\
&\leq C' V_p(\omega; s, t)^{1/p} V_q(\rho; s, t)^{1/q}.
\end{aligned}$$

Thus

$$\begin{aligned}
(4.10) \quad & \left| \int_{t_i}^{t_{i+1}} \rho d\omega - \rho(s_i)(\omega(t_{i+1}) - \omega(t_i)) \right| \\
&\leq C V_q(\rho; t_i, t_{i+1})^{1/q} V_p(\omega; t_i, t_{i+1})^{1/p} \\
&\leq C' (V_q(\rho; t_i, t_{i+1}) + V_p(\omega; t_i, t_{i+1})) V_p(\omega; t_i, t_{i+1})^{(1-r)/p}.
\end{aligned}$$

By applying (4.9) and the similar estimate for  $\rho$ , we get

$$\begin{aligned}
& \left| \int_0^1 \rho d\omega - \sum_i \rho(s_i)(\omega(t_{i+1}) - \omega(t_i)) \right| \\
&\leq C (V_q(\rho) + V_p(\omega)) \sup_i V_p(\omega; t_i, t_{i+1})^{(1-r)/p}
\end{aligned}$$

which converges to 0 since  $\omega$  is continuous.  $\square$

REMARK 4.2. In the proof, we have considered separately the arc  $[A, B]$ . Actually,

$$(4.11) \quad \int_{[A, B]} (\rho(\tau^{\nearrow}) - \rho(\tau^{\searrow})) \lambda(d\tau) = \int \rho d\underline{\omega}$$

with

$$(4.12) \quad \underline{\omega}(t) = \inf_{[0, t]} \omega \vee \inf_{[t, 1]} \omega.$$

REMARK 4.3. In the framework of Theorem 4.1, the fact that our integral is a Riemann–Stieltjes integral implies that it is linear with respect to  $\omega$ ; this property was not evident on our definition, since the tree associated to the sum of two paths is not simply related to the trees of the two paths. Actually, we do not know whether the space of  $\omega$  satisfying (4.3) is linear.

REMARK 4.4. Young integrals can also be written as classical integrals on the time interval by means of a completely different technique, namely fractional differential calculus (see [37]).

THEOREM 4.5. *Theorem 4.1 holds for càdlàg paths  $\omega$ , provided  $\rho$  is continuous at times of discontinuity of  $\omega$ .*

PROOF. The tree is associated to a continuous path  $(\omega'(t, x); (t, x) \in \mathbb{G})$ , as it has been explained in Proposition 2.4, and  $\omega'$  has the same variations as  $\omega$ . One can also consider  $\rho'(t, x) = \rho(t)$  which has the same variations as  $\rho$ . Then the left-hand side of (4.4) is the integral for  $\rho'$  and  $\omega'$ , so (4.4) holds true. For the Riemann sums, we modify (4.10) in the previous proof by introducing  $r' < 1$  such that  $1/p + 1/q = 1/r'$ ; then

$$\begin{aligned} & \left| \int_{t_i}^{t_{i+1}} \rho d\omega - \rho(s_i)(\omega(t_{i+1}) - \omega(t_i)) \right| \\ & \leq C(V_q(\rho; t_i, t_{i+1}) + V_p(\omega; t_i, t_{i+1})) \\ & \quad \times V_p(\omega; t_i, t_{i+1})^{(1-r')/p} V_q(\rho; t_i, t_{i+1})^{(1-r')/q}, \end{aligned}$$

so that

$$\begin{aligned} & \left| \int_0^1 \rho d\omega - \sum_i \rho(s_i)(\omega(t_{i+1}) - \omega(t_i)) \right| \\ & \leq C(V_q(\rho) + V_p(\omega)) \sup_i (V_p(\omega; t_i, t_{i+1})^{1/p} V_q(\rho; t_i, t_{i+1})^{1/q})^{1-r'}. \end{aligned}$$

We have to prove that the supremum tends to 0 as the mesh of the subdivision tends to 0. For any  $\varepsilon > 0$ , let us consider

$$J_\varepsilon := \{i; |\Delta\omega(t)| \geq \varepsilon \text{ for some } t_i < t \leq t_{i+1}\}.$$

Then

$$\limsup \sup_{i \notin J_\varepsilon} V_p(\omega; t_i, t_{i+1})^{1/p} \leq \varepsilon,$$

and the number of jumps greater than  $\varepsilon$  is finite, so from the continuity of  $\rho$  at these points,

$$\lim \sup_{i \in J_\varepsilon} V_q(\rho; t_i, t_{i+1})^{1/q} = 0.$$

We deduce the convergence from these two properties.  $\square$

REMARK 4.6. If  $\rho$  and  $\omega$  have common discontinuity times, our integral can still be defined, but the Riemann–Stieltjes approach has to be modified, as in the classical Young work [36].

This theory can be applied to paths of fractional Brownian motions with Hurst parameter  $H > 1/2$ , or to Lévy processes without Brownian part and such that  $|x|^p \wedge 1$  is integrable with respect to the Lévy measure for some  $p < 2$ .

4.2. *Beyond the Young integral.* A limitation of the Young integral concerns its iteration. If  $\omega$  and  $\rho$  have respectively  $p$ - and  $q$ -finite variation for  $1/p + 1/q > 1$  and  $\omega$  is continuous, then we can consider the function

$$x(t) := x_0 + \int_0^t \rho d\omega,$$

and (4.6) implies that  $x$  has  $p$ -finite variation; however, it generally does not have  $q$ -finite variation so, unless  $p < 2$ , one cannot construct  $\int x d\omega$ . Nevertheless, we now check that this is possible with our framework (for a continuous one-dimensional path  $\omega$ ). The idea is to look for a weaker condition than  $V_q(\rho) < \infty$  for (4.3).

For instance, if  $\rho(t) = f(\omega(t))$ , (4.3) holds for any bounded  $f$  and any continuous  $\omega$ , and

$$\int_0^1 f(\omega(t)) d\omega(t) = F(\omega(1)) - F(\omega(0))$$

for a primitive function  $F$  of  $f$ ; this is because the integral on  $\mathbb{T} \setminus [A, B]$  in (4.2) is 0 ( $f(\omega(\tau^\nearrow)) = f(\omega(\tau^\searrow))$ ), and the integral on  $[A, B]$  is easily computed from (4.11). However, in this case, the integral is not always the limit of Riemann sums, as it is easily seen for  $f(x) = x$ . We want to generalize this example.

Define

$$V_q(\rho|\omega) := \sup \sum_k |\rho(t_{2k+2}) - \rho(t_{2k+1})|^q$$

where the supremum is with respect to subdivisions  $(t_i)$  of  $[0, 1]$  such that  $\omega(t_{2k+1}) = \omega(t_{2k+2})$ , and put

$$\mathcal{V}(\rho|\omega) := \inf \{q \geq 1; V_q(\rho|\omega) < \infty\} \leq \mathcal{V}(\rho).$$

THEOREM 4.7. *Let  $\omega$  be continuous. The integrability condition (4.3) holds as soon as*

$$1/\mathcal{V}(\omega) + 1/\mathcal{V}(\rho|\omega) > 1.$$

Moreover, if  $1/p + 1/q > 1$  and  $\omega$  fixed with  $V_p(\omega) < \infty$ , the space of bounded functions  $\rho$  such that  $V_q(\rho|\omega) < \infty$  is a Banach space  $\mathbb{B}_{q,\omega}$  for the norm

$$\|\rho\|_{q,\omega} := V_q(\rho|\omega)^{1/q} + \sup |\rho|,$$

and we have

$$(4.13) \quad V_q\left(\int_0^\cdot \rho d\omega \mid \omega\right)^{1/q} \leq C V_p(\omega)^{1/p} V_q(\rho|\omega)^{1/q},$$

$$(4.14) \quad \left\| \int_0^\cdot \rho d\omega \right\|_{q,\omega} + V_p\left(\int_0^\cdot \rho d\omega\right)^{1/p} \leq C V_p(\omega)^{1/p} \|\rho\|_{q,\omega},$$

for some  $C = C(p, q)$ .

PROOF. In the estimation (4.7), we can use  $V_q(\rho|\omega)$  instead of  $V_q(\rho)$  since we consider the subdivisions defined by  $t_{2k+1} = \tau^{\nearrow}$  and  $t_{2k+2} = \tau^{\searrow}$  for  $\tau \in \partial\mathbb{T}^a$ . Thus (4.4) is replaced by

$$(4.15) \quad \int_{\mathbb{T}} |\rho(\tau^{\nearrow}) - \rho(\tau^{\searrow})| \lambda(d\tau) \leq C V_p(\omega)^{1/p} \|\rho\|_{q,\omega}.$$

This proves the first statement. The Banach property is easily verified from the lower semicontinuity of  $\rho \mapsto V_q(\rho|\omega)$  with respect to uniform convergence. By applying (4.15) on  $[s, t]$ , we estimate  $\int_s^t \rho d\omega$ , and deduce that  $\int_0^1 \rho d\omega$  and  $V_p(\int \rho d\omega)^{1/p}$  are bounded by the right-hand side of (4.15) [for the estimation of the  $p$ -variation, we use (4.9)]. The last property which has to be proved in order to conclude is (4.13). To this end, we are going to check that

$$(4.16) \quad \left| \int_0^1 \rho d\omega \right| \leq C V_p(\omega)^{1/p} V_q(\rho|\omega)^{1/q}$$

as soon as  $\omega(0) = \omega(1)$ ; then (4.13) follows by applying (4.16) on the intervals  $[t_{2k+1}, t_{2k+2}]$  in order to estimate  $V_q(\cdot|\omega)$ . The left-hand side of (4.16) is written as an integral on the tree; the integral on  $\mathbb{T} \setminus [A, B]$  is estimated by the right-hand side of (4.16) as in (4.8); for the integral on  $[A, B]$ , it can be written as

$$\int_{[A, B]} (\rho(\tau^{\nearrow}) - \rho(\tau^{\searrow})) \lambda(d\tau) = \int_{\inf \omega}^{\omega(0)} (\rho(\beta_2(x)) - \rho(\beta_1(x))) dx$$

with

$$\beta_1(x) = \inf\{t; \omega(t) = x\}, \quad \beta_2(x) = \sup\{t; \omega(t) = x\}.$$

This expression is also easily estimated by the right-hand side of (4.16).  $\square$

As an application, we can solve differential equations driven by a multidimensional path, provided all the components of the path but one are smooth enough.

**THEOREM 4.8.** *For  $1/p + 1/q > 1$  and  $q \leq p$ , consider a continuous real-valued map  $\omega$  with finite  $p$ -variation, and let  $\mathbb{B}_{p,q,\omega}$  be the Banach space of functions  $\rho$  such that*

$$\|\rho\|_{p,q,\omega} := V_p(\rho)^{1/p} + V_q(\rho|\omega)^{1/q} + \sup |\rho|$$

*is finite. Consider also a continuous function  $\eta$  with values in  $\mathbb{R}^{d-1}$  and with finite  $q$ -variation, and let  $\xi = (\omega, \eta)$  with values in  $\mathbb{R}^d$ . Let  $f$  be a  $C^2$  function with bounded derivatives from  $\mathbb{R}^n$  into the space of linear maps  $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$ . Consider, for  $x_0$  in  $\mathbb{R}^n$ , the equation*

$$x(t) = x_0 + \int_0^t f(x(s)) d\xi(s)$$



where the integral should be understood as the sum of integrals with respect to each component, each one being given by an expression of type (4.1) or (4.2). Then this equation has a unique solution in the Banach space  $\mathbb{B}_{p,q,\omega}^n$ .

PROOF. In this proof, the constants  $C$  may depend on  $f$  and  $x_0$ , but not on  $\xi$ . It is not difficult to deduce from the Lipschitz property of  $f$  that

$$F : (x(t); 0 \leq t \leq 1) \mapsto (f(x(t)); 0 \leq t \leq 1)$$

maps  $(\mathbb{B}_{p,q,\omega})^n$  into  $(\mathbb{B}_{p,q,\omega})^{nd}$  and has at most linear growth:

$$(4.17) \quad \|F(x)\|_{p,q,\omega} \leq C(\|x\|_{p,q,\omega} + 1).$$

Let us prove that  $F$  is locally Lipschitz. It is easy to verify

$$(4.18) \quad \sup |F(x_2) - F(x_1)| \leq C \sup |x_2 - x_1|,$$

and let us estimate  $V_q(F(x_2) - F(x_1)|\omega)$ . Let  $(t_i)$  be a subdivision satisfying  $\omega(t_{2k+1}) = \omega(t_{2k+2})$ , and use the notation  $\Delta_i v = v(t_{i+1}) - v(t_i)$ . It follows from the boundedness of the derivatives of  $f$  that

$$\begin{aligned} & |f(x_2(t_{i+1})) - f(x_1(t_{i+1})) - f(x_2(t_i)) + f(x_1(t_i))| \\ & \leq C(|x_2(t_{i+1}) - x_1(t_{i+1})| + |x_2(t_i) - x_1(t_i)|) \\ & \quad \times (|\Delta_i x_2| + |\Delta_i x_1|) + C|\Delta_i x_2 - \Delta_i x_1| \\ & \leq 2C \sup |x_2 - x_1| (|\Delta_i x_2| + |\Delta_i x_1|) + C|\Delta_i x_2 - \Delta_i x_1|. \end{aligned}$$

By taking the  $q$ th power and summing over indices  $i = 2k + 1$ , we deduce

$$\begin{aligned} & V_q(F(x_2) - F(x_1) | \omega) \\ (4.19) \quad & \leq C \sup |x_2 - x_1|^q (V_q(x_1|\omega) + V_q(x_2|\omega)) + CV_q(x_2 - x_1|\omega) \\ & \leq C\|x_2 - x_1\|_{p,q,\omega}^q (V_q(x_1|\omega) + V_q(x_2|\omega) + 1). \end{aligned}$$

We prove similarly that

$$(4.20) \quad V_p(F(x_2) - F(x_1)) \leq C\|x_2 - x_1\|_{p,q,\omega}^p (V_p(x_1) + V_p(x_2) + 1).$$

It follows from (4.18), (4.19) and (4.20) that  $F$  is locally Lipschitz; more precisely,

$$(4.21) \quad \|F(x_2) - F(x_1)\|_{p,q,\omega} \leq C(\|x_1\|_{p,q,\omega} + \|x_2\|_{p,q,\omega} + 1)\|x_2 - x_1\|_{p,q,\omega}.$$

On the other hand, the property  $q \leq p$  and Theorem 4.1 (applied with an exchange of  $p$  and  $q$ ) show that

$$\left\| \int_0^\cdot \rho d\eta \right\|_{p,q,\omega} \leq C \sup \left| \int_0^\cdot \rho d\eta \right| + CV_q \left( \int_0^\cdot \rho d\eta \right)^{1/q} \leq C' \|\rho\|_{p,q,\omega} V_q(\eta)^{1/q}$$

if  $\rho$  takes its values in  $\mathcal{L}(\mathbb{R}^{d-1}, \mathbb{R}^n)$ . If now  $\rho$  takes its values in  $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$ , we deduce by using also (4.14) that

$$(4.22) \quad \left\| \int_0^\cdot \rho d\xi \right\|_{p,q,\omega} \leq C \|\rho\|_{p,q,\omega} (V_p(\omega)^{1/p} + V_q(\eta)^{1/q}).$$

Thus, by joining (4.17), (4.21) and (4.22), we obtain that the map

$$\Phi : (\rho(t); 0 \leq t \leq 1) \mapsto \left( x_0 + \int_0^t \rho d\xi; 0 \leq t \leq 1 \right)$$

satisfies

$$\|(\Phi \circ F)(x)\|_{p,q,\omega} \leq C + C(V_p(\omega)^{1/p} + V_q(\eta)^{1/q})(1 + \|x\|_{p,q,\omega})$$

and

$$\begin{aligned} & \|(\Phi \circ F)(x_2) - (\Phi \circ F)(x_1)\|_{p,q,\omega} \\ & \leq C(V_p(\omega)^{1/p} + V_q(\eta)^{1/q})(\|x_1\|_{p,q,\omega} + \|x_2\|_{p,q,\omega} + 1)\|x_2 - x_1\|_{p,q,\omega}. \end{aligned}$$

It is then classical to deduce that  $\Phi \circ F$  has a unique fixed point if  $V_p(\omega)$  and  $V_q(\eta)$  are small enough. We conclude like for usual differential equations by dividing  $[0, 1]$  into subintervals where  $\omega$  and  $\eta$  have small variation.  $\square$

In particular, we can work out a calculus for one-dimensional fractional Brownian motions of any Hurst parameter, and the stochastic integrals can be interpreted as integrals on the tree; another interpretation can be worked out by modifying Russo–Vallois integrals [14, 28].

*4.3. Integration for fractional Brownian motion.* Up to now, we have found sufficient conditions ensuring that the integral  $\int \rho d\omega$  can be defined as an integral on the tree. However, by means of the disintegration  $\lambda = \lambda_2$  of the length measure (Proposition 2.1), the strong integrability condition (4.3) can be replaced by the weaker condition

$$(4.23) \quad \int \left| \sum_{\tau \in \partial \mathbb{T}^a} (\rho(\tau^{\nearrow}) - \rho(\tau^{\nwarrow})) \right| da < \infty$$

(where the number of terms in the sum is finite), and in this case we can define

$$(4.24) \quad \int_0^1 \rho d\omega := \int \sum_{\tau \in \partial \mathbb{T}^a} (\rho(\tau^{\nearrow}) - \rho(\tau^{\nwarrow})) da$$

[with a form similar to (4.2) if  $\pi(0) \neq \pi(1)$ ]. This is a generalization of the previous framework, and the integral, when it exists, is again the limit of  $\int \rho d\omega^a$ . If (4.23) is satisfied for  $\rho$  replaced by 0 out of  $[s, t]$ , we can define

similarly  $\int_s^t \rho d\omega$  satisfying the Chasles relation. Our aim is now to check that this integral is well adapted to the differential calculus with respect to a finite-dimensional  $H$ -fractional Brownian motion, for  $1/3 < H \leq 1/2$  (made of independent one-dimensional fractional Brownian motions), and that the integrals coincide with those of the rough paths theory [5, 20, 21, 22, 23]. Some related results for the standard Brownian case  $H = 1/2$  are also given in [30]; in this case, the integrals which are considered here are Stratonovich integrals, but it is also explained in [30] how one can use the tree  $\mathbb{T}$  to obtain Itô integrals. We are going to consider the two-dimensional case (higher dimension is similar).

**THEOREM 4.9.** *Consider a two-dimensional  $H$ -fractional Brownian motion for  $H \leq 1/2$ . Then almost any path  $(\omega, \eta)$  satisfies the following properties:*

1. *Suppose  $H > 1/4$  and let  $1/4 < r < H$ . Then the integral  $\int_s^t \eta d\omega$  can be defined in the sense of (4.24). Moreover*

$$(4.25) \quad \gamma(s, t) := \int_s^t \eta d\omega - \eta(s)(\omega(t) - \omega(s))$$

*satisfies*

$$|\gamma(s, t)| \leq K(t - s)^{2r},$$

*where  $K$  depends on  $r$  and the path  $(\omega, \eta)$ , but not on  $(s, t)$ .*

2. *Suppose  $H > 1/3$  and let  $1/3 < r < H$ . Let  $\rho, \phi$  and  $\psi$  be bounded paths such that*

$$(4.26) \quad |\rho(t) - \rho(s) - \phi(s)(\omega(t) - \omega(s)) - \psi(s)(\eta(t) - \eta(s))| \leq K_1(t - s)^{2r}$$

*and*

$$|\psi(t) - \psi(s)| \leq K_2(t - s)^r$$

*for any  $s < t$  [where  $K_1$  and  $K_2$  may depend on  $(\omega, \eta)$ ]. Then the integral  $\int_s^t \rho d\omega$  can be defined in the sense of (4.24), and*

$$(4.27) \quad \left| \int_s^t \rho d\omega - \rho(s)(\omega(t) - \omega(s)) - \frac{\phi(s)}{2}(\omega(t) - \omega(s))^2 - \psi(s)\gamma(s, t) \right| \leq K_3(t - s)^{3r}.$$

**PROOF.** Let  $\mathbb{E}^\omega$  denote the integration with respect to the law of  $\eta$ , with  $\omega$  fixed, and let  $\mathbb{T}$  be the tree of  $\omega$ . We divide the proof of the two parts of the theorem into two steps.

STEP 1. Define a process  $U^a$  as follows: consider the points  $\tau_1, \tau_2, \dots$  of  $\partial\mathbb{T}^a$  such that  $[\tau_i^{\nearrow}, \tau_i^{\nwarrow}] \subset [0, 1]$ , and let  $U^a$  be 0 before  $\tau_1^{\nearrow}$ , be constant on each  $[\tau_i^{\nwarrow}, \tau_{i+1}^{\nearrow}]$  and after the last  $\tau_i^{\nwarrow}$ , be affine on each  $[\tau_i^{\nearrow}, \tau_i^{\nwarrow}]$ , and have the same increment on this interval as  $\eta$ . We will use the notation  $\Delta\tau_i = \tau_i^{\nwarrow} - \tau_i^{\nearrow}$ . Since the increments of  $\eta$  are negatively correlated, we have for  $j \leq k$  and  $\varepsilon$  small enough

$$\begin{aligned} \mathbb{E}^\omega (U^a(\tau_k^{\nwarrow}) - U^a(\tau_j^{\nearrow}))^2 &\leq C \sum_{i=j}^k (\Delta\tau_i)^{2H} \\ &\leq C \left( \inf_i \Delta\tau_i \right)^{-2H+2\varepsilon} \sum_{i=j}^k (\Delta\tau_i)^{4H-2\varepsilon} \\ &\leq C \left( \inf_i \Delta\tau_i \right)^{-2H+2\varepsilon} \left( \sum_{i=j}^k \Delta\tau_i \right)^{4H-2\varepsilon} \\ &\leq K a^{-2+2\varepsilon} (\tau_k^{\nwarrow} - \tau_j^{\nearrow})^{4H-2\varepsilon}, \end{aligned}$$

with  $K = K(\omega)$  bounded in the spaces  $L^q$ ; in the last line, we have used the modulus of continuity of  $\omega$ . Thus  $U^a$  is Hölder continuous in  $L^2(\mathbb{P}^\omega)$  on the set of times  $\{\tau_j^{\nearrow}, \tau_j^{\nwarrow}\}$ ; since it is extended to  $[0, 1]$  by affine interpolation, it satisfies the same property on the whole interval, so

$$\mathbb{E}^\omega (U^a(t) - U^a(s))^2 \leq K a^{-2+2\varepsilon} (t - s)^{4H-2\varepsilon}.$$

Since the variable is conditionally Gaussian, estimates in  $L^q(\mathbb{P}^\omega)$  can be deduced for any  $q$ , so that, after integration with respect to  $\omega$ ,

$$\|U^a(t) - U^a(s)\|_{L^q} \leq C_q a^{-1+\varepsilon} (t - s)^{2H-\varepsilon}.$$

By applying the Kolmogorov lemma,

$$(4.28) \quad |U^a(t) - U^a(s)| \leq K^a a^{-1+\varepsilon} (t - s)^{2r}$$

with  $K^a$  bounded in  $L^q$ , uniformly in  $a$ , and for  $1/4 < r < H - \varepsilon/2$ . Moreover,

$$I_{s,t}^a := \sum_{\tau \in \partial\mathbb{T}^a : [\tau^{\nearrow}, \tau^{\nwarrow}] \subset [s, t]} (\eta(\tau^{\nwarrow}) - \eta(\tau^{\nearrow}))$$

is an increment of  $U^a$  on a subinterval of  $[s, t]$ , so

$$|I_{s,t}^a| \leq K^a a^{-1+\varepsilon} (t - s)^{2r}.$$

Since  $K^a$  is bounded in  $L^1$ ,  $\int_0^{a_0} K^a a^{-1+\varepsilon} da$  is finite for any  $a_0$  and almost any  $(\omega, \eta)$ ; moreover,  $I_{s,t}^a$  is 0 if  $a$  is greater than the oscillation of  $\omega$ . Thus

$$(4.29) \quad \int_0^\infty |I_{s,t}^a| da \leq K (t - s)^{2r}$$

for some finite variable  $K$ . This implies that the integral  $\int_s^t \eta d\omega$  is well defined as claimed in the theorem, and

$$\int_s^t \eta d\omega = \int_0^\infty I_{s,t}^a da + \int_s^t \eta d\underline{\omega}$$

where  $[0, 1]$  is replaced by  $[s, t]$  in the notation (4.12). The estimation of  $\gamma(s, t)$  follows from (4.29) and the moduli of continuity of  $\omega$  and  $\eta$ .

STEP 2. Let us now consider the integral of  $\rho$ . As in the previous step, we consider the term  $K^a$  of (4.28), and a path  $(\omega, \eta)$  such that  $\int_0^{a_0} K^a a^{-1+\varepsilon} da$  is finite. Consider as in the previous step the times  $\tau_j$  of  $\partial\mathbb{T}^a$ , and define  $\psi^a$  to be  $\psi_{\tau_j^\nearrow}$  on each  $[\tau_j^\nearrow, \tau_{j+1}^\nearrow)$ , and 0 before  $\tau_1^\nearrow$ ; then, by limiting the sums to indices  $j$  such that  $[\tau_j^\nearrow, \tau_j^\searrow] \subset [s, t]$ ,

$$\begin{aligned} J_{s,t}^a &:= \sum_j (\rho(\tau_j^\searrow) - \rho(\tau_j^\nearrow)) \\ (4.30) \quad &= \int_{s'}^{t'} \psi^a dU^a + \sum_j (\rho(\tau_j^\searrow) - \rho(\tau_j^\nearrow) - \psi(\tau_j^\nearrow)(\eta(\tau_j^\searrow) - \eta(\tau_j^\nearrow))) \end{aligned}$$

where  $s'$  and  $t'$  are the first  $\tau_j^\nearrow$  and the last  $\tau_j^\searrow$  in  $[s, t]$ . Since  $1/r + 1/(2r) > 1$ , the first term is estimated as a Young integral by means of (4.5), so

$$\begin{aligned} \left| \int_{s'}^{t'} \psi^a dU^a \right| &\leq C V_{1/(2r)}(U^a)^{2r} (V_{1/r}(\psi^a)^r + \sup |\psi^a|) \\ (4.31) \quad &\leq K K^a a^{-1+\varepsilon} (t-s)^{2r} \end{aligned}$$

for a finite  $K$ , and for  $K^a$  obtained in the previous step. The second term of (4.30) is dominated from (4.26) by

$$(4.32) \quad \sum (\tau_j^\searrow - \tau_j^\nearrow)^{2r} \leq K a^{-1+\varepsilon} \sum (\tau_j^\searrow - \tau_j^\nearrow)^{3r} \leq K a^{-1+\varepsilon} (t-s)^{3r}$$

where we have used the modulus of continuity of  $\omega$  in the first inequality. Thus, by adding (4.31) and (4.32), the expression  $J_{s,t}^a$  of (4.30) is integrable with respect to  $a$ , and  $\int_s^t \rho d\omega$  is defined. Moreover,

$$\int_s^t \rho d\omega = \int_s^t \rho d\underline{\omega} + \int J_{s,t}^a da.$$

If  $\rho(s) = \phi(s) = \psi(s) = 0$ , then  $\rho$  is at most of order  $(t-s)^{2r}$ , so the first term is at most of order  $(t-s)^{3r}$ ; on the other hand, in this case, one can put the exponent  $3r$  instead of  $2r$  in (4.31), so the integral of  $J_{s,t}^a$  is also of order  $(t-s)^{3r}$ ; thus  $\int_s^t \rho d\omega$  is of order  $(t-s)^{3r}$ . This can be applied to the integral of

$$\rho(\cdot) - \rho(s) - \phi(s)(\omega(\cdot) - \omega(s)) - \psi(s)(\eta(\cdot) - \eta(s)),$$

and we deduce (4.27).  $\square$

REMARK 4.10. The estimate (4.27) shows that the integral can be constructed by time discretization as limits of generalized Riemann sums

$$(4.33) \quad \int_0^1 \rho d\omega = \lim \sum_i \left( \rho(t_i)(\omega(t_{i+1}) - \omega(t_i)) + \frac{\phi(t_i)}{2}(\omega(t_{i+1}) - \omega(t_i))^2 + \psi(t_i)\gamma(t_i, t_{i+1}) \right).$$

In the framework of Theorem 4.9, we can construct similarly integrals with respect to  $\eta$  by means of the tree of  $\eta$ . Let  $(e_1, e_2)$  be the canonical basis of  $\mathbb{R}^2$ . Put  $\xi = (\omega, \eta)$  and

$$(4.34) \quad \begin{aligned} \Gamma(s, t) &:= \int_s^t (\xi(u) - \xi(s)) \otimes d\xi(u) \\ &= \frac{(\omega(t) - \omega(s))^2}{2} e_1 \otimes e_1 + \frac{(\eta(t) - \eta(s))^2}{2} e_2 \otimes e_2 \\ &\quad + \left( \int_s^t (\eta(u) - \eta(s)) d\omega(u) \right) e_2 \otimes e_1 \\ &\quad + \left( \int_s^t (\omega(u) - \omega(s)) d\eta(u) \right) e_1 \otimes e_2. \end{aligned}$$

It is easy to check that  $\Gamma$  is multiplicative [see the definition in (A.7)], and we obtain a rough path  $(\xi, \Gamma)$ . Moreover, Theorem 4.9 enables to consider integrals with respect to  $\xi$ , and, by applying (4.33) and Theorem A.5, we see that they coincide with the integrals of Appendix A.2, so they match the rough paths theory.

PROPOSITION 4.11. *Let  $\xi$  be a two-dimensional  $H$ -fractional Brownian motion for  $1/3 < H \leq 1/2$ , and let  $\Gamma$  be defined by (4.34). Then the rough path  $(\xi, \Gamma)$  coincides with the rough path constructed by Coutin and Qian [5] by means of linear interpolation on dyadic subdivisions.*

PROOF. It is sufficient to check that the integral  $\gamma(s, t)$  of (4.25) coincides with the other approach, and actually, we only consider  $\gamma(0, 1) = \int_0^1 \eta d\omega$ . For  $\omega$  fixed, the integral  $\int \eta d\omega$  is in the Gaussian space generated by  $\eta$ , so it is characterized by its covariance with the variables  $\eta(t)$ . But, for  $\omega$  fixed,  $\int \eta d\omega^a$  converges in  $L^2$  to  $\int \eta d\omega$ , so

$$\mathbb{E}^\omega \left[ \eta(t) \int_0^1 \eta d\omega \right] = \lim_a \int_0^1 \mathbb{E}[\eta(t)\eta(s)] d\omega^a(s)$$

$$\begin{aligned}
&= \lim_a \int_0^1 (\omega^a(1) - \omega^a(s)) \frac{\partial}{\partial s} \mathbb{E}[\eta(t)\eta(s)] ds \\
&= \int_0^1 (\omega(1) - \omega(s)) \frac{\partial}{\partial s} \mathbb{E}[\eta(t)\eta(s)] ds.
\end{aligned}$$

Thus the integral is in the closed subspace of  $L^2$  generated by the variables  $\omega(u)\eta(t)$ , and is characterized by

$$\begin{aligned}
(4.35) \quad \mathbb{E} \left[ \omega(u)\eta(t) \int_0^1 \eta d\omega \right] &= \int_0^1 \mathbb{E}[\omega(u)(\omega(1) - \omega(s))] \frac{\partial}{\partial s} \mathbb{E}[\eta(t)\eta(s)] ds \\
&= \int_0^1 \mathbb{E}[\eta(t)\eta(s)] \frac{\partial}{\partial s} \mathbb{E}[\omega(u)\omega(s)] ds.
\end{aligned}$$

On the other hand, the Coutin–Qian integral  $\int \eta d_{CQ}\omega$  is also in this closed subspace, and is characterized by

$$(4.36) \quad \mathbb{E} \left[ \omega(u)\eta(t) \int_0^1 \eta d_{CQ}\omega \right] = \lim_n \int_0^1 \mathbb{E}[\eta(t)\eta^n(s)] \frac{\partial}{\partial s} \mathbb{E}[\omega(u)\omega^n(s)] ds,$$

where  $(\omega^n, \eta^n)$  are dyadic approximations of  $(\omega, \eta)$ . We have to prove that the two expressions in (4.35) and (4.36) match. It is clear that the expectations in (4.36) converge, and we can conclude by standard techniques as soon as we prove that

$$(4.37) \quad \sup_n \int_0^1 \left| \frac{\partial}{\partial s} \mathbb{E}[\omega(u)\omega^n(s)] \right|^{1+\varepsilon} ds < \infty$$

for some  $\varepsilon > 0$ . But  $s \mapsto \mathbb{E}[\omega(u)\omega^n(s)]$  is the dyadic approximation of  $s \mapsto \mathbb{E}[\omega(u)\omega(s)]$  which contains two terms ( $s^{2H}$  and  $|u-s|^{2H}$ ) depending on  $s$  (the term  $u^{2H}$  disappears in the differentiation). If  $\{s^{2H}\}^n$  and  $\{|u-s|^{2H}\}^n$  denote their dyadic approximations, then

$$\left| \frac{\partial}{\partial s} \{s^{2H}\}^n \right| \leq s^{2H-1}, \quad \left| \frac{\partial}{\partial s} \{|u-s|^{2H}\}^n \right| \leq |u-s|^{2H-1},$$

so (4.37) holds provided  $(1+\varepsilon)(1-2H) < 1$ .  $\square$

**REMARK 4.12.** It is known from the construction of [5] that the rough path  $(\xi, \Gamma)$  is geometric (it is the limit in  $p$ -variation of finite variation paths with their double integrals). However, we do not know whether it is the limit of  $(\omega^a, \eta^a)$  with its double integrals.

## APPENDIX

**A.1. A mixing property.** We give a result about the long-range dependence of increments of a fractional Brownian motion. This result was used in Proposition 3.7 but may also be of independent interest. After this work was completed, a similar result was proved in [27] with a more functional analytic method.

THEOREM A.1. Consider a fractional Brownian motion  $(W_t; t \in \mathbb{R})$  with parameter  $0 < H < 1$ ; for  $-\infty \leq s \leq t \leq +\infty$ , denote by  $\mathcal{F}_t^s$  the  $\sigma$ -algebra generated by the increments  $W_v - W_u$ ,  $s < u \leq v < t$ . Let  $t_0 < t_1 < t_2$ , let  $F$  and  $G$  be real variables which are respectively measurable with respect to  $\mathcal{F}_{t_0}^{-\infty}$  and  $\mathcal{F}_{t_2}^{t_1}$ , and let  $q > 1$ . We suppose that  $F$  and  $G$  are in  $L^q$ . Let

$$R(t_0, t_1, t_2) := \left( \frac{t_2 - t_1}{t_1 - t_0} \right)^{1-H}.$$

Then if  $R(t_0, t_1, t_2)$  is small enough, the product  $FG$  is integrable and

$$|\mathbb{E}[FG] - \mathbb{E}[F]\mathbb{E}[G]| \leq C \|F\|_q \|G\|_q R(t_0, t_1, t_2)$$

for some  $C = C(q, H)$ .

In particular, for  $q = 2$ , we get an upper bound for the correlation coefficient

$$\rho(\mathcal{F}_{t_0}^{-\infty}, \mathcal{F}_{t_2}^{t_1}) := \sup \left\{ \frac{|\text{cov}(F, G)|}{\sqrt{\text{var}(F) \text{var}(G)}}; F \in \mathcal{F}_{t_0}^{-\infty}, G \in \mathcal{F}_{t_2}^{t_1} \right\}.$$

This bound is valid if  $R(t_0, t_1, t_2)$  is small enough, but the coefficient is of course bounded by 1 everywhere. We deduce that the mixing property

$$(A.1) \quad \rho(\mathcal{F}_{j\delta}^{-\infty}, \mathcal{F}_{(k+1)\delta}^{k\delta}) \leq \frac{C_H}{1 + |k - j|^{1-H}}$$

holds for any  $\delta > 0$  and any integers  $j \leq k$  in  $\mathbb{Z}$ .

REMARK A.2. The order of magnitude claimed in the theorem is optimal, as it can be seen by taking for  $F$  and  $G$  some increments of  $W$ . However, in Proposition 3.7, we do not use the whole  $\sigma$ -algebra  $\mathcal{F}_{j\delta}^{-\infty}$ , but only  $\mathcal{F}_{j\delta}^{(j-1)\delta}$ ; in this case, our estimate is rough but sufficient for our result.

REMARK A.3. One can consider the similar problem for the  $\sigma$ -algebra generated by  $W_u$ ,  $s \leq u \leq t$ , instead of the increments of  $W$ . This question is studied in [2], but the result proved there is not sufficient for us.

For the proof of Theorem A.1, let us first introduce some notation concerning fractional calculus. The fractional integral operator (or left-sided Riemann–Liouville operator) of order  $\alpha > 0$  is defined by

$$I^\alpha g(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds.$$

It satisfies  $I^{\alpha+\beta} = I^\alpha I^\beta$ , and it coincides with the iterated integral of  $g$  if  $\alpha$  is an integer. Moreover  $I^\alpha$  maps the space  $L^q([0, T])$  into itself, and

$$(A.2) \quad I^\alpha \phi_\beta = \phi_{\alpha+\beta} \quad \text{for } \phi_\beta(t) = t^\beta / \Gamma(\beta + 1), \beta > -1.$$



Consider the fractional Brownian motion  $W$  of the theorem. The result is trivial if  $H = 1/2$ , so we suppose  $H \neq 1/2$ . From the shift invariance, we can also suppose  $t_0 = 0$ . The Mandelbrot–Van Ness definition states that if  $(B_t; t \in \mathbb{R})$  is a double standard Brownian motion, then

$$(A.3) \quad W_t := C \int_{\mathbb{R}} ((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}) dB_s$$

is a fractional Brownian motion for  $C > 0$ ; we will choose the normalization

$$C = C_H := \Gamma(H + 1/2)^{-1}.$$

We also consider an independent standard Brownian motion  $(\bar{B}_t; t \leq 0)$ , and we let  $\mathcal{F}_0$  and  $\mathcal{F}'_0$  be the  $\sigma$ -algebras generated respectively by  $(B_s; s \leq 0)$  and  $(B_s, \bar{B}_s; s \leq 0)$ .

LEMMA A.4. *Let  $(f(t); 0 \leq t \leq t_2 - t_1)$  be a random function which is measurable with respect to  $\mathcal{F}'_0$  and such that  $f(0) = 0$ . We suppose that  $f = I^{H+1/2}g$  for a function  $g$  in  $L^2([0, t_2 - t_1])$ . Consider the perturbed process*

$$(A.4) \quad \widetilde{W}_t := W_t + f(t - t_1)1_{\{t \geq t_1\}}, \quad t \leq t_2.$$

*Then, if  $G(W)$  is a functional depending (as in Theorem A.1) on the increments of  $W$  between times  $t_1$  and  $t_2$ ,*

$$\begin{aligned} & |\mathbb{E}[G(W) \mid \mathcal{F}_0] - \mathbb{E}[G(\widetilde{W}) \mid \mathcal{F}_0]| \\ & \leq C \mathbb{E}[|G(\widetilde{W})|^q \mid \mathcal{F}_0]^{1/q} \mathbb{E}[(L^{1/2}e^{CL})^p \mid \mathcal{F}_0]^{1/p} \end{aligned}$$

*for  $1/p + 1/q = 1$  and some  $C = C(q)$ , and with*

$$L := \int_0^{t_2 - t_1} g(s)^2 ds.$$

PROOF. By definition, we have

$$f(t - t_1)1_{\{t \geq t_1\}} = C_H \int_{t_1}^{t \vee t_1} (t - s)^{H-1/2} g(s - t_1) ds,$$

so

$$\widetilde{W}_t = C_H \int_{\mathbb{R}} ((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}) d\widetilde{B}_s$$

with

$$\widetilde{B}_t = B_t + \int_{t_1}^{t \vee t_1} g(s - t_1) ds.$$

The process  $B$  is perturbed after time  $t_1$  by an absolutely continuous process which is  $\mathcal{F}'_0$ -measurable, so by writing the Cameron–Martin theorem conditionally on  $\mathcal{F}'_0$ ,

$$\begin{aligned} \mathbb{E}[G(W) \mid \mathcal{F}'_0] \\ = \mathbb{E}\left[G(\widetilde{W}) \exp\left(-\int_{t_1}^{t_2} g(s-t_1) dB_s - \frac{1}{2} \int_{t_1}^{t_2} g(s-t_1)^2 ds\right) \mid \mathcal{F}'_0\right]. \end{aligned}$$

By conditioning on  $\mathcal{F}_0 \subset \mathcal{F}'_0$ ,

$$\mathbb{E}[G(W) \mid \mathcal{F}_0] - \mathbb{E}[G(\widetilde{W}) \mid \mathcal{F}_0] = \mathbb{E}[G(\widetilde{W})(\exp(\cdots) - 1) \mid \mathcal{F}_0]$$

and the result follows from Hölder's inequality and standard estimates on the moments of  $\exp(\cdots) - 1$ .  $\square$

PROOF OF THEOREM A.1. We use previous notation, and in particular suppose  $H \neq 1/2$  and  $t_0 = 0$ . Define

$$(A.5) \quad f(t) := C_H \int_{-\infty}^0 ((t+t_1-s)^{H-1/2} - (t_1-s)^{H-1/2})(d\overline{B}_s - dB_s)$$

for  $0 \leq t \leq t_2 - t_1$ . Let us assume that  $f$  satisfies the assumption of Lemma A.4 (this will be proved later). Consider the process  $\widetilde{W}$  of (A.4), and the process  $\overline{W}$  obtained from  $W$  by replacing  $B$  by  $\overline{B}$  on  $(-\infty, 0]$  in (A.3), so that

$$\overline{W}_t = C_H \int_{-\infty}^{t \wedge 0} ((t-s)^{H-1/2} - (-s)^{H-1/2}) d\overline{B}_s + C_H \int_{t \wedge 0}^t (t-s)^{H-1/2} dB_s.$$

Then  $\overline{W}$  has the same law as  $W$ , is independent from  $\mathcal{F}_0$ , and

$$\widetilde{W}_{t+t_1} - \widetilde{W}_{t_1} = W_{t+t_1} - W_{t_1} + f(t) = \overline{W}_{t+t_1} - \overline{W}_{t_1},$$

so  $G(\widetilde{W}) = G(\overline{W})$  is independent from  $\mathcal{F}_0$  and has the same law as  $G = G(W)$ . Thus we can use

$$\mathbb{E}[G(\widetilde{W}) \mid \mathcal{F}_0] = \mathbb{E}[G], \quad \mathbb{E}[|G(\widetilde{W})|^q \mid \mathcal{F}_0]^{1/q} = \|G\|_q$$

in Lemma A.4, so that

$$|\mathbb{E}[G \mid \mathcal{F}_0] - \mathbb{E}[G]| \leq C \|G\|_q \mathbb{E}[(L^{1/2} e^{CL})^p \mid \mathcal{F}_0]^{1/p}.$$

Thus

$$\begin{aligned} |\text{cov}(F, G)| &\leq C \|G\|_q \mathbb{E}[|F| \mathbb{E}[(L^{1/2} e^{CL})^p \mid \mathcal{F}_0]^{1/p}] \\ &\leq C \|G\|_q \|F\|_q \|L^{1/2} e^{CL}\|_p. \end{aligned}$$

In order to conclude, we have to estimate this  $L^p$  norm. The formula (A.5) for  $f$  can be differentiated, so  $f$  is smooth and

$$f^{(k)}(t) = \frac{1}{\Gamma(H-k+1/2)} \int_{-\infty}^0 (t+t_1-s)^{H-k-1/2} (d\overline{B}_s - dB_s)$$

for  $k \geq 1$ . In particular,

$$\|f^{(k)}(t)\|_r = C(t + t_1)^{H-k}$$

for any  $r$  and some  $C = C(r, k, H)$ . On the other hand [recall the definition of  $\phi_\beta$  in (A.2)],

$$f = f'(0)\phi_1 + I^2(f'') = I^{H+1/2}g$$

for

$$g = f'(0)\phi_{1/2-H} + I^{3/2-H}(f'').$$

In particular,  $f$  satisfies the assumption of Lemma A.4. Moreover,

$$\|f'(0)\|_r \phi_{1/2-H}(t) = Ct_1^{H-1}t^{1/2-H},$$

and

$$\begin{aligned} \|I^{3/2-H}(f'')(t)\|_r &\leq C \int_0^t (t-s)^{1/2-H} \|f''(s)\|_r ds \\ &\leq C' \int_0^t (t-s)^{1/2-H} (t_1+s)^{H-2} ds \\ &\leq C'' t_1^{H-1} t^{1/2-H} \end{aligned}$$

where the last estimate is easily obtained by considering separately the integrals on  $[0, t/2]$  and  $[t/2, t]$ . Thus we have obtained an estimate for  $\|g(t)\|_r$ , and we deduce that

$$\|L^{1/2}\|_r \leq C((t_2 - t_1)/t_1)^{1-H} = CR(0, t_1, t_2)$$

for any  $r$  and some  $C = C(r, H)$ . We still have to prove that the moments of  $\exp(L)$  are bounded; but, from Jensen's inequality,

$$\exp(rL) \leq \frac{1}{\mathbb{E}L} \int_0^{t_2-t_1} \exp\left(r \frac{g(s)^2}{\mathbb{E}g(s)^2} \mathbb{E}L\right) \mathbb{E}[g(s)^2] ds,$$

so, since  $g(s)$  is Gaussian, this expression has bounded expectation provided  $r\mathbb{E}L < 1/2$ , and therefore if  $R(t_0, t_1, t_2)$  is small enough.  $\square$

**A.2. Rough paths.** Our aim is to describe a part of the rough paths theory through a point of view which is well adapted to our approach (Theorem 4.9). Our result (Theorem A.5 below) is in particular comparable to [13, 15, 20], and we include for completeness a short proof which is sufficient for our purpose. Let  $\xi(t)$  be a path with finite  $p$ -variation, for  $p < 3$ . In this case, we learn from the theory of rough paths that  $\xi$  is not sufficient for the construction of an integral calculus, but we also need its double integrals.

More precisely, let  $\xi(t)$  and  $\Gamma(s, t)$  take their values respectively in  $\mathbb{R}^d$  and  $\mathbb{R}^d \otimes \mathbb{R}^d$ . We suppose that

$$(A.6) \quad |\xi(t) - \xi(s)| \leq \mu(t - s)^r, \quad |\Gamma(s, t)| \leq \mu(t - s)^{2r}$$

for  $r = 1/p$  (continuous paths with finite  $p$ -variation can be reduced to this case by a change of time). The path is supposed to be multiplicative in the sense

$$(A.7) \quad \Gamma(s, t) = \Gamma(s, u) + \Gamma(u, t) + (\xi(u) - \xi(s)) \otimes (\xi(t) - \xi(u))$$

for  $s \leq u \leq t$ . If  $r > 1/2$ , then  $\Gamma$  is necessarily the Young integral

$$\Gamma(s, t) = \int_s^t (\xi(u) - \xi(s)) \otimes d\xi(u),$$

but if  $1/3 < r \leq 1/2$ , the function  $\Gamma$ , when it exists, is not unique; one can add to it  $\phi(t) - \phi(s)$  for any  $(2r)$ -Hölder continuous  $\phi$ . Let us now explain how one can define integrals  $\int \rho d\xi$ , in a way which coincides with the tree approach of Theorem 4.9.

**THEOREM A.5.** *Consider paths  $(\xi, \Gamma)$  satisfying (A.6) and (A.7),  $\rho$  with values in  $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$  (the space of linear maps), and  $\Phi$  with values in the space  $\mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)) = \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^n)$ . We suppose that*

$$|\rho(t) - \rho(s) - \Phi(s)(\xi(t) - \xi(s))| \leq \mu'(t - s)^{2r}$$

and

$$|\Phi(t) - \Phi(s)| \leq \mu'(t - s)^r.$$

For any  $s < t$  and any subdivision  $\Sigma = (t_k)$  of  $[s, t]$ , put

$$(A.8) \quad g(\Sigma) := \sum_k (\rho(t_k)(\xi(t_{k+1}) - \xi(t_k)) + \Phi(t_k)\Gamma(t_k, t_{k+1})).$$

Then  $g(\Sigma)$  converges as  $\max(t_{k+1} - t_k)$  tends to 0, and the limit  $\int_s^t \rho d\xi$  satisfies

$$(A.9) \quad \left| \int_s^t \rho d\xi - \rho(s)(\xi(t) - \xi(s)) - \Phi(s)\Gamma(s, t) \right| \leq C\mu\mu'(t - s)^{3r}$$

for some  $C = C(r)$ .

**REMARK A.6.** The identification  $\mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)) = \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^n)$  is made through  $[G(x)](y) = G(x \otimes y)$ .

REMARK A.7. We use the simple notation  $\int \rho d\xi$  though the integral actually depends on  $(\rho, \Phi)$  and  $(\xi, \Gamma)$ . Notice, however, that if

$$\limsup_{t \downarrow s} |\xi(t) - \xi(s)| / (t - s)^{2r} = +\infty$$

for almost any  $s$  (and this is the case for an  $H$ -fractional Brownian motion and  $1/3 < r < H \leq 1/2$ ), then  $\Phi$  is uniquely determined by  $\rho$ .

PROOF OF THEOREM A.5. In the proof we will use the following result taken from Young integration. Let  $g(\Sigma)$  be a function defined on finite subdivisions  $\Sigma = (t_k)$  of  $[s, t]$  and let  $\Sigma_k$  be the subdivision with  $t_k$  removed. We suppose that

$$(A.10) \quad |g(\Sigma) - g(\Sigma_k)| \leq C_g(t_{k+1} - t_{k-1})^\kappa$$

for some  $\kappa > 1$ . Then  $g(\Sigma)$  converges as the mesh of  $\Sigma$  tends to 0, and

$$|\lim g - g(o)| \leq C(\kappa)C_g(t - s)^\kappa$$

where the trivial subdivision  $o = (s, t)$ . Let  $g$  be the functional of (A.8). Then

$$\begin{aligned} g(\Sigma) - g(\Sigma_k) &= \rho(t_{k-1})(\xi(t_k) - \xi(t_{k-1})) + \Phi(t_{k-1})\Gamma(t_{k-1}, t_k) \\ &\quad + \rho(t_k)(\xi(t_{k+1}) - \xi(t_k)) + \Phi(t_k)\Gamma(t_k, t_{k+1}) \\ &\quad - \rho(t_{k-1})(\xi(t_{k+1}) - \xi(t_{k-1})) - \Phi(t_{k-1})\Gamma(t_{k-1}, t_{k+1}) \\ &= (\rho(t_k) - \rho(t_{k-1}))(\xi(t_{k+1}) - \xi(t_k)) \\ &\quad - \Phi(t_{k-1})(\xi(t_k) - \xi(t_{k-1})) \otimes (\xi(t_{k+1}) - \xi(t_k)) \\ &\quad + (\Phi(t_k) - \Phi(t_{k-1}))\Gamma(t_k, t_{k+1}) \end{aligned}$$

where we have used the multiplicative property of  $\Gamma$ . The condition (A.10) is satisfied with  $\kappa = 3r$  and  $C_g = 2\mu\mu'$ , so the result is proved.  $\square$

In particular, we can compute the integral  $\int f(\xi) d\xi$  of a one-form by considering  $\rho = f(\xi)$  and  $\Phi = f'(\xi)$ ; the property (A.9) implies that the integral is the limit of generalized Riemann sums, so it coincides with the standard rough paths approach.

## REFERENCES

- [1] ALDOUS, D. (1993). The continuum random tree. III. *Ann. Probab.* **21** 248–289. [MR1207226](#)
- [2] BERKES, I. and HORVÁTH, L. (1999). Limit theorems for logarithmic averages of fractional Brownian motions. *J. Theoret. Probab.* **12** 985–1009. [MR1729465](#)
- [3] BERTOIN, J. (1996). *Lévy Processes. Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. [MR1406564](#)

- [4] BRUNEAU, M. (1979). Sur la  $p$ -variation d'une surmartingale continue. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78). Lecture Notes in Math.* **721** 227–232. Springer, Berlin. [MR544794](#)
- [5] COUTIN, L. and QIAN, Z. (2002). Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theory Related Fields* **122** 108–140. [MR1883719](#)
- [6] DUQUESNE, T. (2006). The coding of compact real trees by real valued functions. Preprint.
- [7] DUQUESNE, T. and LE GALL, J.-F. (2002). Random trees, Lévy processes and spatial branching processes. *Astérisque* 281.
- [8] DUQUESNE, T. and LE GALL, J.-F. (2005). Probabilistic and fractal aspects of Lévy trees. *Probab. Theory Related Fields* **131** 553–603. [MR2147221](#)
- [9] DUQUESNE, T. and LE GALL, J.-F. (2006). The Hausdorff measure of stable trees. *ALEA Lat. Am. J. Probab. Math. Stat.* **1** 393–415 (electronic). [MR2291942](#)
- [10] EVANS, S. N. (2008). *Probability and Real Trees. Lecture Notes in Math.* **1920**. Springer, Berlin. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005. [MR2351587](#)
- [11] EVANS, S. N., PITMAN, J. and WINTER, A. (2006). Rayleigh processes, real trees, and root growth with re-grafting. *Probab. Theory Related Fields* **134** 81–126. [MR2221786](#)
- [12] FALCONER, K. (1990). *Fractal Geometry*. John Wiley & Sons Ltd., Chichester. Mathematical foundations and applications. [MR1102677](#)
- [13] FEYEL, D. and DE LA PRADELLE, A. (2006). Curvilinear integrals along enriched paths. *Electron. J. Probab.* **11** 860–892 (electronic). [MR2261056](#)
- [14] GRADINARU, M., NOURDIN, I., RUSSO, F. and VALLOIS, P. (2005).  $m$ -order integrals and generalized Itô's formula: The case of a fractional Brownian motion with any Hurst index. *Ann. Inst. H. Poincaré Probab. Statist.* **41** 781–806. [MR2144234](#)
- [15] GUBINELLI, M. (2004). Controlling rough paths. *J. Funct. Anal.* **216** 86–140. [MR2091358](#)
- [16] ISTAS, J. and LANG, G. (1997). Quadratic variations and estimation of the local Hölder index of a Gaussian process. *Ann. Inst. H. Poincaré Probab. Statist.* **33** 407–436. [MR1465796](#)
- [17] KESTEN, H. (1986). Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. H. Poincaré Probab. Statist.* **22** 425–487. [MR871905](#)
- [18] LE GALL, J.-F. (1991). Brownian excursions, trees and measure-valued branching processes. *Ann. Probab.* **19** 1399–1439. [MR1127710](#)
- [19] LE GALL, J.-F. and LE JAN, Y. (1998). Branching processes in Lévy processes: The exploration process. *Ann. Probab.* **26** 213–252. [MR1617047](#)
- [20] LEJAY, A. (2003). An introduction to rough paths. In *Séminaire de Probabilités XXXVII. Lecture Notes in Math.* **1832** 1–59. Springer, Berlin. [MR2053040](#)
- [21] LYONS, T. J. (1998). Differential equations driven by rough signals. *Rev. Mat. Iberoamericana* **14** 215–310. [MR1654527](#)
- [22] LYONS, T. J., CARUANA, M. and LÉVY, T. (2007). *Differential Equations Driven by Rough Paths. Lecture Notes in Math.* **1908**. Springer, Berlin. Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004, With an introduction concerning the Summer School by Jean Picard. [MR2314753](#)
- [23] LYONS, T. and QIAN, Z. (2002). *System Control and Rough Paths. Oxford Mathematical Monographs*. Oxford Univ. Press, Oxford. Oxford Science Publications. [MR2036784](#)

- [24] MOLCHAN, G. M. (1999). Maximum of a fractional Brownian motion: Probabilities of small values. *Comm. Math. Phys.* **205** 97–111. [MR1706900](#)
- [25] NEVEU, J. (1986). Erasing a branching tree. *Adv. in Appl. Probab.* suppl. 101–108. [MR868511](#)
- [26] NEVEU, J. and PITMAN, J. (1989). Renewal property of the extrema and tree property of the excursion of a one-dimensional Brownian motion. In *Séminaire de Probabilités XXIII. Lecture Notes in Math.* **1372** 239–247. Springer, Berlin. [MR1022914](#)
- [27] NORROS, I. and SAKSMAN, E. (2007). Local independence of fractional Brownian motion. Preprint.
- [28] NOURDIN, I. (2008). A simple theory for the study of SDEs driven by a fractional Brownian motion, in dimension one. In *Séminaire de Probabilités XLI. Lecture Notes in Math.* **1934**. Springer, Berlin.
- [29] PERES, Y. (1999). Probability on trees: An introductory climb. In *Lectures on Probability Theory and Statistics (Saint-Flour, 1997). Lecture Notes in Math.* **1717** 193–280. Springer, Berlin. [MR1746302](#)
- [30] PICARD, J. (2006). Brownian excursions, stochastic integrals, and representation of Wiener functionals. *Electron. J. Probab.* **11** 199–248 (electronic). [MR2217815](#)
- [31] PISIER, G. and XU, Q. H. (1988). The strong  $p$ -variation of martingales and orthogonal series. *Probab. Theory Related Fields* **77** 497–514. [MR933985](#)
- [32] PITMAN, J. (2006). *Combinatorial Stochastic Processes. Lecture Notes in Math.* **1875**. Springer, Berlin. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard. [MR2245368](#)
- [33] RUSSO, F. and VALLOIS, P. (1993). Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields* **97** 403–421. [MR1245252](#)
- [34] STRICKER, C. (1979). Sur la  $p$ -variation des surmartingales. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78). Lecture Notes in Math.* **721** 233–237. Springer, Berlin. [MR544795](#)
- [35] WILLIAMS, D. R. E. (2001). Path-wise solutions of stochastic differential equations driven by Lévy processes. *Rev. Mat. Iberoamericana* **17** 295–329. [MR1891200](#)
- [36] YOUNG, L. C. (1936). An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.* **67** 251–282. [MR1555421](#)
- [37] ZÄHLE, M. (1998). Integration with respect to fractal functions and stochastic calculus. I. *Probab. Theory Related Fields* **111** 333–374. [MR1640795](#)

LABORATOIRE DE MATHÉMATIQUES (CNRS UMR 6620)  
 UNIVERSITÉ BLAISE PASCAL  
 63177 AUBIÈRE CEDEX  
 FRANCE  
 E-MAIL: [Jean.Picard@math.univ-bpclermont.fr](mailto:Jean.Picard@math.univ-bpclermont.fr)